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S_n-SUBMODULES OF FREE NOVIKOV ALGEBRAS

Annotation. Algebra with identities, $a \circ (b \circ c) - (a \circ b) \circ c = a \circ (c \circ b) - (a \circ c) \circ b$ $a \circ (b \circ c) = b \circ (a \circ c)$ is called Novikov. Studied two S_n-submodules of free Novikov algebra and decomposition of them into Specht modules.

Keywords: Novikov algebras, Specht modules.

Ключевые слова: алгебра Новикова, модуль Шпехта.

Тірек сөздер: Новиков алгебрасы, Шпехт модулі.

1. Introduction. In 1950, A.I. Malcev [9] and W. Specht [11] independently used the representation theory of symmetric group to classify polynomial identities of algebraic structures. These module structures are known for many classes of algebras and well used in classifying varieties of algebras. S_n – module structure of free associative algebras, Leibniz, Zinbiel and Poisson algebras is regular module. S_n – module structure of Lie algebras is studied by Klyachko [8], Kraskiewicz ajjd Weynman [7]. In [1] studied varieties of anti-commuative algebras. S_n – module structure of bicommuative algebras is studied in [4]. In our paper we are introduced in decomposition of some S_{n+1} – submodules of free Novikov algebras into irreducible modules (Specht modules).

Let us introduce non-associative non-commutative polynomials of degree three *rsym* (right-symmetric polynomial) and *lcom* (left-commutative polynomial) by

$$rsym = t_1(t_2t_3) - t_1(t_3t_2) - (t_1t_2)t_3 + (t_1t_3)t_2,$$

$$lcom = t_1(t_2t_3) - t_2(t_1t_3).$$

An algebra with identities $rsym = 0$ and $lcom = 0$ is called *right-Novikov*. Since we consider in our paper only right-Novikov algebras the word "right" will be omitted. So, if $A = (A,\circ)$ is a Novikov algebra with multiplication $a \circ b$, then

$$(a,b,c) = (a.c.b),$$

$$a \circ (b \circ c) = b \circ (a \circ c)$$

for any $a, b, c \in A$. Here

$$(a,b,c) = a \circ (b \circ c) - (a \circ b) \circ c$$

is associator.

Example 1.1. *Let* $A = C[x]$ *and* $a \circ b = \partial(a)b$, *where* $\frac{\partial}{\partial x}$ *be partial derivation. Then* $(A,\circ)$ *is Novikov algebra.*

2. Partition and base of free Novikov algebra. In [2], [3] there was constructed base of free Novikov algebras in terms of rooted trees and so-called *r*-elements and in terms of partitions. For our future consideration, we use base in terms of partitions.

Let $n \in N$ natural number and $P(n)$ is a set of partitions of n such that

$$P(n) = \{\alpha = (\alpha_1,\ldots,\alpha_k) | \alpha_1 + \cdots + \alpha_k = n, \alpha_1 \geq \cdots \alpha_k \geq 1, 1 \leq k \leq n\}$$

Shortly, we write $\alpha \vdash n$. Let Y_α is a Young diagram of a form α, i.e. a diagram with α_i boxes in the *i*-th row. There is a one-to-one correspondence between partitions of n and Young diagrams of order n.

For a partition $\alpha \vdash n$ denote by $\tilde{\alpha} = (\tilde{\alpha}_1,\ldots,\tilde{\alpha}_k)$ a partition of n such that „ $\tilde{\alpha}_1 = \alpha_1 + 1$, „ $\tilde{\alpha}_i = \alpha_i$ $i > 1$. $Y_{\tilde{\alpha}}$ is called a Novikov diagram. There is one-to-one correspondence between Young diagrams of order n and Novikov diagrams of order $n+1$

$$Y_\alpha = \begin{array}{cccccc} \bullet & \cdots & \bullet & \bullet & \bullet \\ \bullet & \cdots & \bullet & \bullet & \\ \vdots & \cdots & \vdots & \vdots & \\ \bullet & \cdots & \bullet & & \end{array} \qquad Y_{\tilde{\alpha}} = \begin{array}{cccccc} \bullet & \cdots & \bullet & \bullet & \bullet & \circ \\ \bullet & \cdots & \bullet & \bullet & & \\ \vdots & \cdots & \vdots & \vdots & & \\ \bullet & \cdots & \bullet & & & \end{array}$$

So, a Novikov diagram is obtained by Young diagram by adding to the first row one box. Call this box "nose". Present $Y_{\tilde{\alpha}}$ as disjoint union of two parts,

$$Y_{\tilde{\alpha}} = Y_{\tilde{\alpha}l} \cup Y_{\tilde{\alpha}r},$$

where $Y_{\tilde{\alpha}l}$, call it *left* part of $Y_{\tilde{\alpha}}$, is left column of $Y_{\tilde{\alpha}}$, and, $Y_{\tilde{\alpha}r}$, call it *right* part of $Y_{\tilde{\alpha}}$, is a complement of $Y_{\tilde{\alpha}l}$ in $Y_{\tilde{\alpha}}$.

$$
\begin{array}{l|l}
|Y_{\tilde{\alpha}l}\| & Y_{\tilde{\alpha}r} \quad | \\
\hline
| \bullet \| & \cdots \bullet \;\; \bullet \;\; \bullet \;\; \circ \;\; | \\
| \bullet \| & \cdots \bullet \;\; \bullet | \\
| \vdots \| & \cdots \vdots \;\; \vdots | \\
| \bullet \| & \cdots \bullet |
\end{array}
$$

Now we have to fill Novikov diagram $Y_{\tilde{\alpha}}$ by elements of Ω. Filling of boxes by elements of alphabet depends whether this box is on the left part or on the right part . Denote by $f_{i,j}$ an element of Ω in the box $(i.j)$ that is a cross of i -th row by j -th column. The fillling rule is the following

- , $f_{i,1}$. $\geq f_{i,i+1}$., if , $\alpha_i = \alpha_{i+1}$., i=1,2,…,k-1.
- the sequence $f_{k,2} \cdots f_{k,\alpha_k} f_{k-1,2} \cdots f_{k-1,\alpha_{k-1}} \cdots f_{1,2} \cdots f_{1,\alpha_1}, f_{1,\alpha_1+1}$ is non-decreasing.

Such filling of Novikov diagramm we call *Novikov filling.* Call obtained tableaux as Novikov tableau with shape and (Novikov) filling *f.* Denote it by $Y_{\tilde{\alpha} f}$.

Let us given Novikov tableau $Y_{\tilde{\alpha} f}$,

$$
\begin{array}{l}
f_{1,1} f_{1,2} \cdots\cdots\cdots f_{1\alpha_1} \; f_{1,\alpha_1+1} \\
f_{2,1} f_{2,2} \cdots\cdots f_{2\alpha_2} \\
\vdots \quad \vdots \cdots \vdots \\
f_{k,1} f_{k,2} \; \cdots \; f_{k,\alpha_k}
\end{array}
$$

Then base element of free Novikov algebra constructed by $Y_{\tilde{\alpha},f}$, is

$$e_{\alpha,f} = X_{k,f} \circ (\ldots (X_{2,f} \circ X_{1,f}) \ldots),$$

where

$$X_{i,f} = (\ldots (f_{i,1} \circ f_{i,2}) \ldots) \circ f_{i,\alpha_i}, k \geq i > 1,$$

$$X_{1,f} = ((\ldots (f_{1,1} \circ f_{1,2}) \ldots) \circ f_{1,\alpha_1}) \circ f_{1,\alpha_1+1}.$$

So, any base element $e_{\alpha,f}$ of free Novikov algebra of degree $n + 1$ can be characterized by a partition

$\alpha \vdash n$ and by a filling f. For partitions $\alpha = \{ \alpha_1, \alpha_2, \ldots \alpha_k\} \vdash n$ and $\beta = \{ \beta_1, \beta_2, \ldots \beta_l\} \vdash n$ we say that α

dominates β, and write $\alpha \trianglerighteq \beta$, if

$$\alpha_1+\alpha_2+\ldots+\alpha_i \geq \beta_1+\beta_2+\ldots+\beta_i$$

for all $i\geq 1$. If $i>k$ (respectively, $i > l$), then we take α_i (respectively, β_i) to be zero. Recall that

for α, $\beta \vdash n$, the Kostka number $K_{\alpha\beta}$ is equal to the number of semistandard Young tableaux of shape α

and content β. Specht modules corresponding to a partition α denote S^{α}. For more details see [5], [6].

3. Permutation modules and free Novikov algebras. Let $F^{(n+1)}$ be multilinear part of free Novikov algebra generated by *n+1* elements $a_1,\ldots,a_{n+1}$. Consider $F^{(n+1)}$ as S_{n+1} - modules with a natural action

$$\sigma X(a_1, \ldots, a_{n+1}) = X\left(a_{\sigma(1)}, \ldots, a_{\sigma(n+1)}\right).$$

Let $X = \sum_{\vartheta\in Base_{n+1}} \lambda_\vartheta \vartheta$ be an element of $F_{(n+1)}$, where $\lambda_v \in K$. Say that X has *degree* α and write

$deg(X) = \alpha$, if $\lambda_u = 0$ as soon as $deg\, u > \alpha$ and $\lambda_\vartheta \neq 0$, for some $\vartheta \in Base_{n+1}$ with $deg(\vartheta) = \alpha$.

Note that

$$degX < degY \rightarrow X<Y.$$

Proposition 3.1. *For any* X, Y $\in F^{(n+1)}$

$$deg\,(X + Y) < max(deg\, X, deg\, Y),$$

$$deg\,(\lambda X) < degX,\ \lambda \in K.$$

Proof. Let

$$X = \sum_{f_1} \lambda_{f_1} e_{\gamma 1}, f_1 + \sum_{f_2} \lambda_{f_2} e_{\gamma 2}, f_2 + \cdots + \sum_{f_l} \lambda_{f_l} e_{\gamma l}, f_l$$

and

$$Y = \sum_{g_1} \lambda_{g_1} e_{\delta 1}, g_1 + \sum_{g_2} \lambda_{g_2} e_{\delta 2}, g_2 + \cdots + \sum_{g_m} \lambda_{g_m} e_{\delta m}, g_m,$$

such that $\gamma_1 > \gamma_2 > \cdots > \gamma_l$, $\delta_1 > \delta_2 > \cdots > \delta_m$ and $\lambda_{f1} \neq 0.\ \lambda_{g1} \neq 0$ for some f_1 *and* g_1 . Then

$$deg\ X = \gamma_1; \quad deg\ Y = \delta_1$$

if *degX<degY,* then

$$\max(\deg X, \deg Y) = \delta_1 = \deg (X + Y).$$

Similarly, if *degY < deg X,* then

$$max(degX, deg\ Y) = \gamma_1 = \deg (X + Y).$$

If *degX = degY and X≠-Y,* then

$$max(degX, degY) = \delta_1 \geq \deg (X + Y).$$

If *X= -Y*, then

$$deg(X + Y) = deg\ 0 < \delta_1 = max(deg\ X, deg\ Y).$$

So, for any *X and Y*,

$$deg\ (X + Y) \leq max(degX, deg\ Y).$$

It is easy see that if $\lambda \neq 0$, then $deg\ (\lambda X) = deg\ X$, if $\theta = 0$ then , $deg\ (\theta X) \leq deg\ X$.

Now we are able to construct filtration on $F^{(n+1)}$. By Proposition 3.1 for any $\alpha \vdash n$ set of elements of degree no more than α forms linear subspace. Let F_α subspace of $F^{(n+1)}$ generated by all base elements of a form $e_{\beta,f}$ such that α≥β. One can easy show that $F_{(1^n)}$ is a S_{n+1}

Theorem 3.2. $F_{(2.1^{n-2})}$ and $F_{(1^n)}$are S_{n+1} − modules.

Proof. To prove that F_α is a S_{n+1} module for any $\alpha \vdash n$, it is enough to show that for any transposition o∈ S_{n+1} we have $oe_{\alpha.f} \in F_\alpha$. Then it follows $F_{(1^n)}$ is an S_{n+1}- module.

Suppose that $e_{(2.1^{n-2}),f} = c_1 o \left(\ldots \left(c_{n-2}\ o\ \left((a_1 o b_1) o b_1 \right) \right) \ldots \right)$ and o is a transposition.

If σ acts on $c_1, \ldots, c_{n-2}$, then by left-commutativity rule, we obtain again $e_{(2.1^{n-2}),f}$.

If σ acts on b_1, b_2 then

$$\sigma e_{(2,1^{n-2}),f} = c_1 \circ (\cdots (a_{n-2} \circ ((a_1 \circ b_2) \circ b_1)) \cdots) =$$

(by right-symmetric rule)

$$c_1 o \left(\ldots \left(c_{n-2} o (a_1 o (b_2 o b_1)) \right) \ldots \right) - c_1 o \left(\ldots \left(c_{n-2} o (a_1 o (b_1 o b_2)) \right) \ldots \right) + c_1 o \left(\ldots \left(c_{n-2}\ o\ \left((a_1 o b_1) o b_2 \right) \right) \ldots \right)$$

By using left-commutativity rule we obtain element of $F_{(2.1^{n-2})}$. Any other actions of o on $e_{(2.1^{n-2}),f}$ gives element of $F_{(2.1^{n-2})}$.

Theorem 3.3. *There hold following* S_{n+1} *- modules isomophisms*

a. $F_{(1^n)} \cong S^{(n+1)} \oplus S^{(n,1)}$

$if\ n > 3, then\ F_{(2.1^{n-2})} \cong$

b. $2S^{(n+1)} \oplus \quad 3S^{(n,1)} \oplus S^{(n-1.1^2)} \oplus 2S^{(n-1.2)} \oplus S^{(n,1)} \oplus S^{(n-2.3)} \oplus S^{(n-2.2.1)}$.

Proof. We decompose $F_{(1^n)}$ and $F_{(2.1^{n-2})}$ into Specht modules, first we show that $F_{(1^n)}$ is isomorphic to permutation module $M^{(n,1)}$ and $F_{(2.1^{n-2})}/F_{(1^n)}$ is isomorphic to $M^{(n-2.2.1)}$ as S_{n+1} - modules. See [10] more details about permutations modules. Let us express elements of $F_{(1^n)}$ as Novikov tableau:

$$e_{(1^n),f} = a_1 \circ (\cdots (a_{n-1} \circ (a_n \circ a_{n+1}))\cdots) = \begin{matrix} a_n & a_{n+1} \\ a_{n-1} & \\ \vdots & \\ a_1 & \end{matrix}$$

and we have

$$\begin{matrix} a_{\sigma(n)} & a_{n+1} \\ a_{\sigma(n-1)} & \\ \vdots & \\ a_{\sigma(1)} & \end{matrix} = \begin{matrix} a_n & a_{n+1} \\ a_{n-1} & \\ \vdots & \\ a_1 & \end{matrix}$$

for any $\sigma \in S_{n+1}$. If we consider elements $e_{(1^n),f}$ in the following form

$$\begin{matrix} a_1 \cdots & a_{n-1} & a_n \\ a_{n+1} & & \end{matrix}$$

then we obtain an S_{n+1}–module structure which is isomorphic to per-mutation module $M^{(n,1)}$ corresponding to partition (n,1). Then by using we obtain Young's $F_{(1^n)} \cong S^{n+1} \oplus S^{(n,1)}$. Now, express elements of $F_{(2.1^{n-2})}$ as Novikov tableau

$$e_{(2.1^{n-2}),f} = a_1 o(\dots (a_{n-2} o(a_{n-1} o a_n) o a_{n+1})) \dots) = \begin{matrix} a_{n-1} & a_n & a_{n+1} \\ a_{n-2} & & \\ \vdots & & \\ a_1 & & \end{matrix}$$

and we have showed

$$\begin{matrix} a_{n-1} & a_{\tau(n)} & a_{\tau(n+1)} \\ a_{o(n-2)} & & \\ \vdots & & \\ a_{o(1)} & & \end{matrix} = \begin{matrix} a_{n-1} & a_n & a_{n+1} \\ a_{n-2} & & \\ \vdots & & \\ a_1 & & \end{matrix}$$

for any $o, \tau \in S_{n+1}$, If we consider elements $e_{(2.1^{n-2}),f}$ in the following form

$a_1 \ a_2 \ \dots \quad a_{n-3} \quad a_{n-2}$
$a_n \qquad a_{n+1}$
a_{n-2}

then we obtain an S_{n+i}– module structure which is isomorphic to permutation module $M^{(n-2.2.1)}$ corresponding to partition (n-2,2,1). By Young's rule

$$M^{(n-2.2.1)} \cong \bigoplus_{\lambda \triangleright (n-2,2,1)} K_{\lambda(n-2,2,1)} S^{\lambda} .$$

By using the definition of Kostka numbers, we get

$$K_{(n-2,2,1)\ (n-2,2,1)} = K_{(n-2,3)(n-2,2,1)} = K_{(n-1,1,1)} = K_{(n+1)} = 1,$$
$$K_{(n-1,2)\ (n-2,2,1)} = K_{(n,1)(n-2,2,1)} = 2$$

$$\frac{F_{(2,1^{n-2})}}{F_{(1^n)}} \cong S^{n+1} \oplus 2S^{(n,1)} \oplus S^{(n-1,1^2)} \oplus 2S^{(n-1,2)} \oplus S^{(n-2,3)} \oplus S^{(n-2,2,1)}$$

Since $F_{(2,1^{n-2})} = \frac{F_{(2,1^{n-2})}}{F_{(1^n)}} \oplus F_{(1^n)}$,

$$F_{(2,1^{n-2})} \cong 2S^{(n+1)} \oplus 2S^{(n-1,2)} \oplus S^{(n-1,1^2)} \oplus 2S^{(n-1,2)} \oplus S^{(n-2,3)} \oplus S^{(n-2,2,1)}$$

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ЕРКІН НОВИКОВ АЛГЕБРАСЫНЫҢ ІШКІ Sn-МОДУЛДЕРІ

n бүтін санының жіктелуімен анықталатын еркін Новиков алгебрасының кейбір ішкі кеңістіктері зерттелген. Олардың ішкі Sn-модулдері болатындығы дәлелденген және осы модулдерде әрбір Шпехт модулдерінің еселіктері есептелген.

Тірек сөздер: Новиков алгебрасы, Шпехт модулі.

Резюме

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Sn-ПОДМОДУЛИ СВОБОДНОЙ АЛГЕБРЫ НОВИКОВА

Изучаются некоторые подпространства свободной алгебры Новикова, которые определяются разбиением числа n. Доказывается, что они являются Sn-подмодулями и вычисляется кратности каждой Шпехт модули в этих подмодулях.

Ключевые слова: алгебра Новикова, модуль Шпехта.

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