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# SOME TRIPLE OPERADS

**Annotation.** Studied Loday's questions about triple operads when an algebra has Novikov, bicommutative and right-symmetric structures, primtive part has Lie structure of generalized bialgebras. Shown nonexistence of coalgebra structure in case Novikov and bicommutative and given dimension of coalgebra in case right-symmetric.

**Keywords:** triple operads, Novikov algebra, bicommutative algebra, right-symmetric algebra.

**Тірек сөздер:** үштік операдтар, Новиков алгебрасы, бикоммутативті алгебрасы, оң-симметриялы алгебрасы.

**Ключевые слова:** тройка операдов, алгебра Новикова, бикоммутативная алгебра, право-симметрическая алгебра.

**Introduction.** J.-L. Loday introduced in [6] the notion *triple of operads* $(C, \chi, F : A-a\lg \to P-a\lg)$, abbreviated $(C,A,P)$ consisting of operads $C$ and $A$, a compatibility relations $\chi$ between $C^c$-coalgebras and $A$-algebras defining $(C,\chi,A)$-bialgebras, an operad $P$ describing the algebraic structure of the primitive part $\Pr im(H)$ of the bialgebra, and a forgetful functor $F$ from the category of $A$-algebras to the category of $P$-algebras. Let $U$ be a left adjoint to $F$. A triple of operads $(C,A,P)$ is to be the good if the following three conditions are equivalent:

(a) $H$ is connected,

(b) $H \cong U(\Pr im(H))$,

(c) $H$ is cofree over its primitive part.

Operads for associative, commutative (associative) and Lie algebras are denoted by *As*, *Com* and *Lie* respectively. The classical case is $(C,A,P) = (Com, As, Lie)$. Other type of good triple of operads can be found in [6].

Let $A$ is an operad for Lie admissible algebras. J.-L. Loday asked in [6] whether there are an operad $C$ and compatibility relation $\chi$ such that $(C,\chi,A,Lie)$ is a good triple. Novikov, bicommutative and right-symmetric are examples of Lie admissible algebras. $A = (A,\circ)$ is Novikov algebra with multiplication $a \circ b$, if

$$(a,b,c) = (a,c,b),$$
$$a \circ (b \circ c) = b \circ (a \circ c),$$

for any $a,b,c \in A$. Here

$$(a,b,c) = a \circ (b \circ c) - (a \circ b) \circ c$$

is associator.

Let $A = C[x]$ and $a \circ b = \partial(a)b$, where $\partial = \frac{\partial}{\partial x}$ be partial derivation. Then $(A,\circ)$ is Novikov algebra. In [3] and [4] are given construction of base of free Novikov algebra. Algebra with identities

$a(bc) = b(ac$ (left-commutative),

$(ab)c = (ac)b$ (right-commutative)

is called bicommutative. A base, dimension and $S_n$-module structure of bicommutative algebras are given in [5].

Algebra with identity

$$(a,b,c) = (a,c,b)$$

is called right-symmetric. In [2] and [3] are given construction of base of free right-symmetric algebra. Let *Nov*, *Bicom* and $\Re sym$ to be operads for Novikov, bicommutative and right-symmetric algebras respectively.

**Main result.** Let $\Re(n)$ is a $S_n$-module for $\Re$ -algebras generated by n elements and let

$$f^{\Re}(t) = \sum_{n\geq 1} \frac{\dim \Re(n)}{n!} t^n.$$

**Proposition** *([6]). If (C,A,P) is a good triple of operads, then there is a identity of formal power series:*

$$f^A(t) = f^C(f^P(t)).$$

The Stirling numbers of the first and the second kind are denoted by If $S_1(n,k)$ and $S_2(n,k)$ respectively. The unsigned Stirling numbers of the first kind are denoted by $c(n,k)$. Recall that

$$S_1(n,k) = (-1)^{n-k} c(n,k). \tag{1}$$

See [7], for more details about these numbers.

**Theorem. a.** *There is no operad $C_1$, $C_2$ and compatibility relation $\chi_1, \chi_2$, such that $(C_1, \chi_1, Nov, Lie)$, $(C_2, \chi_2, Bicom, Lie)$ are good triples.*

**b.** *If $(C, \Re sym, Lie)$ is a good triple of operads, then*

$$\dim C(n) = \sum_{k=1}^{n} (-1)^{n-k} S_2(n,k) k^{k-1}.$$

**Proof.** Suppose that there is an operad $C_1$ and compatibility relation $\chi_1$ such that $(C_1, \chi_1, Nov, Lie)$ is a good triple. Then by proposition 1.1, we calculate dimension of $C_1$-coalgebras and obtain

$$f^{C_1}(t) = t + \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{3t^4}{4!} + O(x^5).$$

By the some way, we can show for *Bicom*.

To prove the second part of proposition 1.2, it is enough to show that

$$f^{Rsym}(x) = f^C(f^{Lie}(x))$$

where

$$f^{Rsym}(x) = \sum_{n=1} n^{n-2} \frac{t^n}{(n-1)!}, \quad f^{Lie}(x) = -\log(1-x).$$

In our proof we use the formula (3.5.3) in [8]

$$\sum_{n=0}^{\infty} c(n,k) \frac{x^n}{n!} = \frac{(-\log(1-x))^k}{k!} \tag{2}$$

and the proposition 1.4.1 in [7], for all non-negative integers n, k

$$\sum_{m=0}^{\infty} S_1(n,m) S_2(m,k) = \delta_{n,k}. \tag{3}$$

Recall that $S_1(n,k) = 0$ and $S_2(n,k) = 0$ if $n < k$. So, we can write the (3) by

$$\sum_{m=k}^{n} S_1(n,m) S_2(m,k) = \delta_{n,k}. \tag{4}$$

So,

$$f^{Rsym}(x) = f^C(-\log(1-x)) = \sum_{m=1}^{\infty} C(m) \frac{(-\log(1-x))^m}{m!} =$$

(by (2))

$$\sum_{m=1}^{\infty} C(m) \sum_{n=1}^{\infty} c(n,m) \frac{x^n}{n!} = \sum_{n=1}^{\infty} (\sum_{m=1}^{\infty} C(m)c(n,m)) \frac{x^n}{n!}.$$

So, we have to prove that

$$\sum_{m=1}^{\infty} C(m)c(n,m) = n^{n-1}.$$

Here, we use an evident formula for any sequences $a(n,m)$ and $b(n,m)$

$$\sum_{m=1}^{n} a(n,m) \sum_{k=1}^{m} b(m,k) = \sum_{k=1}^{n} \sum_{m=k}^{n} a(n,m)b(m,k). \quad (5)$$

$$\sum_{m=1}^{\infty} C(m)c(n,m) = \sum_{m=1}^{n} c(n,m) \sum_{k=1}^{m} (-1)^{m-k} S_2(m,k)k^{k-1} =.$$

(by (5))

$$\sum_{k=1}^{n} \sum_{m=k}^{n} (-1)^{m-k} c(n,m)S_2(m,k)k^{k-1} =$$

(by (1))

$$\sum_{k=1}^{n} \sum_{m=k}^{n} (-1)^{n-k} S_1(n,m)S_2(m,k)k^{k-1} = \sum_{k=1}^{n} (-1)^{n-k} k^{k-1} \sum_{m=k}^{n} S_1(n,m)S_2(m,k) =$$

(by (4))

$$\sum_{k=1}^{n} (-1)^{n-k} k^{k-1} \delta_{n,k} = n^{n-1}.$$

These numbers also give the number of labeled connected chordal P4-free graphs with $n$ vertices [1] which may be used to describe operads for coalgebra.

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**Резюме**


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КЕЙБІР ҮШТІК ОПЕРАДТАР


Жалпыланған биалгебраның алгебралық құрылымы Новиков, бикоммутативті және оң-симметриялы алгебрасы болғанда, ал примитивті бөлігі Ли алгебрасы болғанда Лоденің үштік операдқа қатысты сұрақтары қарастырылған. Новиков және бикоммутатив алгебралары жағдайында коалгебралық құрылымның жоқтығы көрсетілген және оң-симметриялы алгебра кезінде коалгебраның өлшемі есептелген.



**Тірек сөздер:** үштік операдтар, Новиков алгебрасы, бикоммутативті алгебрасы, оң-симметриялы алгебрасы.

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НЕКОТОРЫЕ ТРОЙКА ОПЕРАДОВ

Изучается вопросы Лоде о тройках операдов если в качестве алгебры рассматриваются алгебру Новикова, бикоммутативную и право-симметрическую алгебру, а в качестве примитивной части рассматривается алгебру Ли обобщенной биалгебры. Показано несуществование коалгебр в случаи алгеброй Новикова и бикоммутативной алгебры и в случае правосимметрической алгебры даны размерность коалгебры.

**Ключевые слова:** тройка операдов, алгебра Новикова, бикоммутативная алгебра, право-симметрическая алгебра.

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