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APPROXIMATE SOLUTION OF ONE, THIRD TYPE BOUNDARY-VALUE PROBLEM FOR THE HEAT EQUATION BY IEF METHOD

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Key words: Integral Error Functions, IEF method
Abbreviations: *IEF-Integral Error Function*

Abstract: Approximate solution of one, third type boundary-value problem is represented by Integral Error Functions method or IEF method, which enables to solve wide range of heat equations with moving boundaries, which degenerate at the initial time.

Introduction

Auto-model case when the boundary $\alpha(t)$ is moving according to the law $\alpha(t) = c\sqrt{t}$ is considered in [1] where analytical solution is found.

Analytical solution of Heat equation in the domain with moving $\beta\sqrt{t} \langle\, x \,\langle a\sqrt{t}$ boundaries *represented in [2].*

Development of methods of solution of free boundary problems is very important for analysis of dynamics of phenomena of heat and mass transfer with phase transformation.

Solution of the Heat Equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

can be represented in the following form

$$u_n(\pm x, t) = t^{\frac{n}{2}} i^n erfc\left(\frac{\pm x}{2a\sqrt{t}}\right) \tag{2}$$

where

$$erfcx = 1 - erfx,\ i^n erfcx = \int_x^\infty i^{n-1} erfcv\,dv,\ \text{n=1,2,...}\ i^0 erfcx \equiv erfcx = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-v^2)dv \tag{3}$$

Using superposition principle solution of (1) can be written in the form of series of (2)

$$u(x,t) = \sum_{n=0}^{k} [A_n u_n(x,t) + B_n u_n(-x,t)], \tag{4}$$

where coefficients A_n, B_n have to be determined.

Finally solution of the heat equation (1) can be represented in the following form

$$u(x,t) = \sum_{n=0}^{k} \left(\sqrt{t}\right)^n \left[A_n i^n erfc\frac{x}{2a\sqrt{t}} + B_n i^n erfc\frac{-x}{2a\sqrt{t}}\right] \tag{5}$$

Using formula for Hermite polynomials one can derive

$$i^n erfc(-x) + (-1)^n i^n erfcx = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2m}}{2^{2m-1} m!(n-2m)!} \tag{6}$$

If $n = 2k$, then

$$i^{2k} erfc\ x + i^{2k} erfc\ (-x) = \sum_{m=0}^{k} \frac{x^{2(k-m)}}{2^{2m-1} m!(2k-2m)!}$$

If $n = 2k + 1$, then

$$i^{2k+1} erfc(-x) - i^{2k+1} erfc\ x = \sum_{m=0}^{k} \frac{x^{2(k-m)+1}}{2^{2m-1} m!(2k-2m+1)!}$$

Then expression (5) can be represented in the following form

$$u(x,t) = \sum_{n=0}^{k} \left\{ A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m} \right\} (7)$$

Problem statement:

Approximate solution of the Heat Equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < t, \quad t > 0$$

Subject to

$$I.C: u(x,o) = 0, \qquad 0 < x < t,$$

$$B.C: \left(u + 2\frac{\partial u}{\partial x}\right)\Big|_{x=0} = e^t, \qquad t > 0, \left(3u + 4\frac{\partial u}{\partial x}\right)\Big|_{x=t} = 1, \quad t > 0.$$

represented in the following form where even and odd coefficients A_{2n}, A_{2n+1} have to be determined.

Substituting expression (7) into the boundary conditions for certain values of

t_k: $t_1 = 0, t_2 = 0.2, t_3 = 0.4, t_4 = 0,6, t_5 = 0,8$ we obtain

For x=0

$$\left(A_0\beta_{0,0} + A_2 t_k \beta_{2,1} + A_4 t_k^2 \beta_{4,2} + \cdots + A_{2k} t_k^{2k} \beta_{2k,k}\right) + 2\{A_1\beta_{1,0} + A_3 t_k \beta_{3,1} + A_5 t_k^2 \beta_{5,2} + \cdots + A_{2k+1} t_k^k \beta_{2k+1,k}\} =$$

$$= e^{t_k}$$

For $x = t$

$$3\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(t_k)^{2n-2m} t_k^m \beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n}(t_k)^{2n-2m+1} t_k^m \beta_{2n+1,m}\right\}$$

$$+ 4\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(2n-2m)(t_k)^{2n-2m-1} t_k^m \beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)(t_k)^{2n-2m} t_k^m \beta_{2n+1,m}\right\} = 1$$

Using Mathcad program it is possible to calculate coefficients from above systems of linear equations.

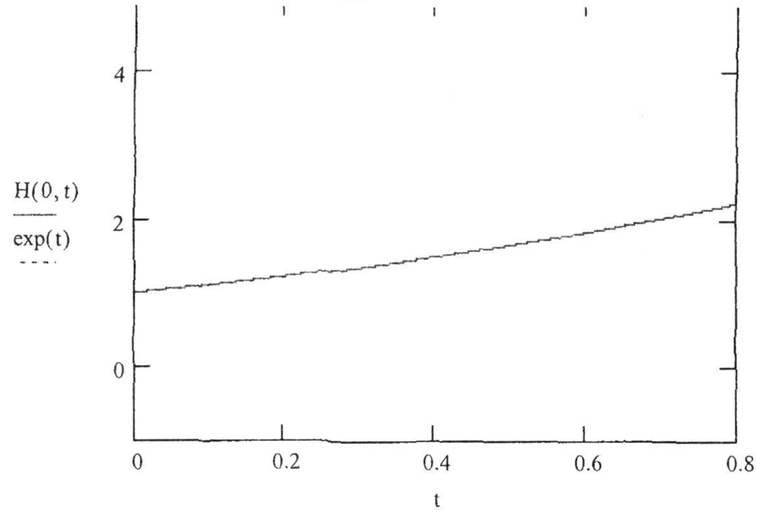

Figure 1. Graphs of functions exp(t) and (7) for x=0

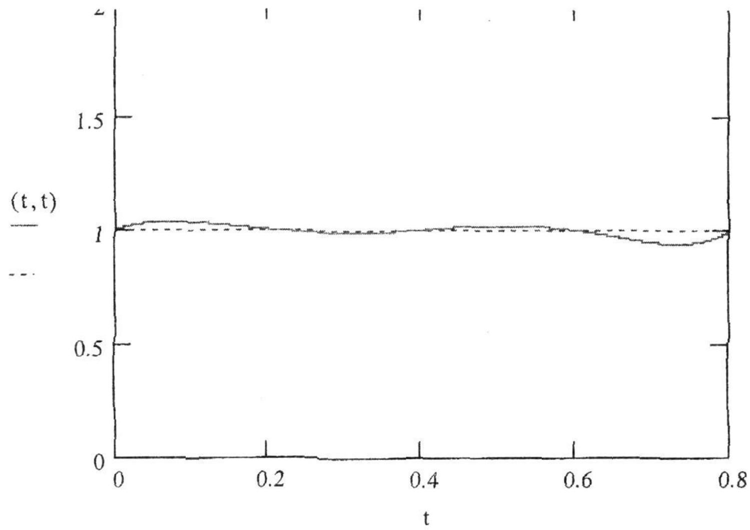

Figure 2. Graphs of constant 1 and function (7) for x=t

Where

$$H(0,t) = \left(A_0\beta_{0.0} + A_2 t_k \beta_{2.1} + A_4 t_k{}^2 \beta_{4.2} + \cdots + A_{2k} t_k{}^{2k} \beta_{2k.k}\right) + 2\{A_1\beta_{1.0} + $$
$$+ A_3 t_k \beta_{3.1} + A_5 t_k{}^2 \beta_{5.2} + \cdots + A_{2k+1} t_k{}^k \beta_{2k+1.k}\},$$

and $\exp(t) = e^{t_k}$

$$\Psi(t,t) = 3\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(t_k)^{2n-2m} t_k{}^m \beta_{2n.m} + A_{2n+1}\sum_{m=0}^{n}(t_k)^{2n-2m+1} t_k{}^m \beta_{2n+1.m}\right\}$$
$$+ 4\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(2n-2m)(t_k)^{2n-2m-1} t_k^m \beta_{2n.m} + \right.$$

$$+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)(t_k)^{2n-2m}t_k^m\beta_{2n+1,m}\Bigg\}$$

It is possible to observe in above figures, that deviation among given boundary functions and functions obtained by substitution expression (7) into the boundary conditions, is less than 10%. Maximum Principle allows us to say that error of the solution doesn't exceed than 10% for 5 points of time *t*. As more precise solution demanded as more values of time *t* should be taken.

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Сәрсенгелдин М.М.

ҮШІНШІ ШЕТКІ ЖЫЛУ ӨТКІЗГІШТІК ТЕҢДЕУІНІҢ ИНТЕГРАЛДЫ ҚАТЕЛІКТЕР ФУНКЦИЯСЫ АРҚЫЛЫ ЖУЫҚ ШЕШІМІ

Электрлік байланыс жүйелерінде жылу үдерістерінің математикалық үлгерін құруға және сипаттауға мүмкіндік беретін үшінші ретті жылу өткізгіштік теңдеулерінің ықтималдық интегралдар функциясы және оның қасиеттері арқылы жуық шешімдері жайында айтылған.

Сарсенгельдин М.М.

ПРИБЛИЖЕННОЕ РЕШЕНИЕ ОДНОЙ ТРЕТЬЕЙ КРАЕВОЙ ЗАДАЧИ МЕТОДОМ ИНТЕГРАЛЬНОЙ ФУНКЦИИ ОШИБОК

Статья посвещена приближенному решению третьей краевой задачи уравнения теплопроводности методом интегральной функции ошибок и ее свойств.