

## The Integral Error Functions Method for solving Heat equation and its application

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**Key words:** Integral Error Functions

**Abstract.** Analytical solution of automodel heat transfer problem is represented by Integral Error Functions method. We observe that proposed method nicely fits the real life problem which is considered in the paper.

**Introduction** It is Hartree 1935 who studied properties of Integral Error Function and reasonably sometimes these functions are called Hartree functions. We follow the method proposed by S.N. Kharin which is represented in [1], [2] and can be effectively used in diverse electric contact phenomena as it was shown in [3], [4].

*Integral Error Functions and its properties*

The integral error functions determined by recurrent formulas

$$i^n \operatorname{erfcx} = \int_x^\infty i^{n-1} \operatorname{erfcx} dv, \quad n=1,2,\dots \quad i^0 \operatorname{erfcx} \equiv \operatorname{erfcx} = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-v^2) dv \quad (1)$$

where

$$\operatorname{erfx} = 1 - \operatorname{erfcx} = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-v^2) dv \quad (2)$$

It is well known that the Integral Error Functions

$$u_n(\pm x, t) = t^{\frac{n}{2}} i^n \operatorname{erfc} \frac{\pm x}{2a\sqrt{t}} \quad (3)$$

exactly satisfy the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (4)$$

and by superposition principle, linear combination of (3) or even series also satisfy (4)

$$u(x, t) = \sum_{n=0}^{\infty} [A_n u_n(x, t) + B_n u_n(-x, t)] \quad (5)$$

We consider (4) and solution (5) in degenerate domain where constants  $A_n, B_n$  have to be determined and can be derived by substituting (5) into boundary conditions if given boundary functions can be expanded into Taylor series with powers  $t$  or  $\sqrt{t}$ .

3. Using formula for Hermite polynomials one can derive

$$i^n \operatorname{erfc}(-x) + (-1)^n i^n \operatorname{erfc}x = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2m}}{2^{2m-1} m! (n-2m)!} \quad (6)$$

and represent (5) in the form of heat polynomials

$$u(x, t) = \sum_{n=0}^{\infty} A_{2n} \sum_{m=0}^n x^{2n-2m} t^{2m} \beta_{2n,m} + A_{2n+1} \sum_{m=0}^n x^{2n-2m+1} t^{2m} \beta_{2n+1,m} \quad (7)$$

where

$$\beta_{n,m} = \frac{1}{2^{n+m-1} \cdot m! \cdot (n-2m)!} \quad (8)$$

4. Using L'Hopital rule it is not difficult to show that

$$\lim_{x \rightarrow \infty} \frac{i^n \operatorname{erfc}(-x)}{x^n} = \frac{2}{n!} \quad (9)$$

### Problem statement

The mathematical model of the temperature distribution in a copper semi-infinite bar with zero initial temperature and the entering heat flux density  $P_0(t) = k + b\sqrt{t}$  where,  $k = 2 \cdot 10^{10} \text{ w} \cdot \text{m}^{-2} \cdot \text{k}^{-1}$ ,

$b = 5 \cdot 10^{11} \text{ w} \cdot \text{m}^{-2} \cdot \text{k}^{-1} \text{sec}^{\frac{1}{2}}$ ,  $a = 9,4 \cdot 10^{-3} \text{ m} \cdot \text{s}^{-0,5}$  and where also the time of melting point has to be found is represented as following automodel heat transfer problem.

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad 0 < x < \infty \quad (10)$$

$$t = 0: \quad u(x, 0) = 0 \quad (11)$$

$$x = 0: \quad -\lambda \frac{\partial u(0, t)}{\partial x} = P_0(t) \quad (12)$$

$$x = \infty: \quad u(\infty, t) = 0 \quad (13)$$

which can be solved by heat potential of single layer

$$u(x, t) = \int_0^t \frac{a e^{-\frac{x^2}{4a^2(t-\tau)}}}{\sqrt{\pi(t-\tau)}} \mu(\tau) d\tau$$

or by any classical method like Laplace transform etc.

**Problem solution:**

We represent solution in the following form:

$$u(x,t) = \sum_{n=0}^{\infty} A_n (2a\sqrt{t})^n i^n \operatorname{erfc}\left(\frac{x}{2a\sqrt{t}}\right) \quad (14)$$

where coefficients  $A_n$  have to be found.

$$u_x(0,t) = \lambda \sum_{n=0}^2 A_n (2a\sqrt{t})^{n-1} i^{n-1} \operatorname{erfc}(0) = P_0(t) \quad (15)$$

$$u_x(0,t) = \frac{\lambda A_0 i^{-1} \operatorname{erfc}(0)}{2a\sqrt{t}} + \lambda A_1 \operatorname{erfc}(0) + \lambda 2a\sqrt{t} A_2 i \operatorname{erfc}(0) = k + b\sqrt{t} \quad (16)$$

$$t^{\frac{1}{2}} : \quad \frac{\lambda A_0 i^{-1} \operatorname{erfc}(0)}{2a\sqrt{t}} = 0 \quad A_0 = 0 \quad (17)$$

$$t^0 : \quad \lambda A_1 \operatorname{erfc}(0) = k \quad A_1 = \frac{k}{\lambda \operatorname{erfc}(0)} \quad (18)$$

$$t^{\frac{1}{2}} : \quad \lambda 2a\sqrt{t} A_2 i \operatorname{erfc}(0) = b\sqrt{t} \quad A_2 = \frac{b}{\lambda 2a i \operatorname{erfc}(0)} \quad (19)$$

$$u(0,t) = \frac{k 2a\sqrt{t}}{\lambda \operatorname{erfc}(0)} i \operatorname{erfc}(0) + \frac{b 2at}{\lambda i \operatorname{erfc}(0)} i^2 \operatorname{erfc}(0) = u_m \quad (20)$$

Rest coefficients of  $A_n$  where  $n > 2$  are equal to zero.

$$\frac{k 2a\sqrt{t}}{\lambda \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{\pi}}} \frac{\Gamma(1)}{\sqrt{\pi}} + \frac{b 2at}{\lambda \frac{\Gamma(1)}{\sqrt{\pi}}} \frac{\Gamma\left(\frac{3}{2}\right)}{2\sqrt{\pi}} = u_m$$

$$\frac{k 2a\sqrt{t}}{\lambda \Gamma\left(\frac{1}{2}\right)} \Gamma(1) + \frac{bat}{\lambda \Gamma(1)} \Gamma\left(\frac{3}{2}\right) = u_m$$

$$\frac{k 2a\sqrt{t}}{\lambda \sqrt{\pi}} + \frac{bat\sqrt{\pi}}{2\lambda} = u_m$$

Let  $t = t_m$  be the time, when the temperature at  $x = 0$  becomes melting point  $u_m$

$$\text{Then } \frac{2ak}{\lambda \sqrt{\pi}} \sqrt{t_m} + \frac{ab\sqrt{\pi}}{2\lambda} t_m = u_m$$

$$\text{or } t_m + 2A\sqrt{t_m} - B = 0$$

$$\text{where } A = \frac{2k}{b\pi}, \quad B = \frac{2\lambda u_m}{ab\sqrt{\pi}}$$

Solving this quadratic equation we get  $\sqrt{t_m} = -A + \sqrt{A^2 + B}$ ,

$$t_m = \left(-A + \sqrt{A^2 + B}\right)^2 \quad (21)$$

For copper

$$\lambda = 300 \text{ w} \cdot \text{m}^{-1} \cdot \text{K}^{-1}, u_m = 1083^\circ\text{C}$$

and we get the value for  $t_m$  :

$$t_m = 2,516 \cdot 10^{-6} \text{ sec} \quad (22)$$

Thus coefficients  $A_n$  of solution function (14) are determined from (17)-(19). Melting time of copper is given by (22).

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#### Метод Интегральных Функций Ошибок для решения уравнения теплопроводности и его приложение

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**Ключевые слова:** Интегральная Функция Ошибок

**Аннотация:** Найдено аналитическое решение уравнения теплопроводности в полубесконечном стержне методом интегральных функций ошибок с помощью которого определяется время плавления меди.

#### Жылуөткізгіштік тендеуінің Интегралды Қателіктер Функциялары арқылы шешімі және қолданбалары

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**Тірек сөздер:** Интегралды Қателіктер Функциясы

**Аннотация:** Жылуөткізгіштік тендеуінің интегралды қателіктер функциялары арқылы шешімі және мыстың балку уақыты табылды.

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