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ANALYTICAL SOLUTION OF HEAT EAQUATION WITH DISCONTINUOUS COEFFICIENTS BY HEAT POLYNOMIALS

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Key words: analytical form, method, integral, heat polynomials.
Abstract. Solution of heat equation with discontinuous coefficients represented in explicit analytical form. The developed method is based on the use of heat polynomials which are derived from Integral Error Functions. This novel method enables to solve heat and mass transfer problems and can be effectively used in the fields of engineering, which require consideration of phenomena with phase transformations, such as low temperature plasma, filtration.The main idea of this method is to find coefficients of heat polynomials which priori satisfy the heat equation.

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АНАЛИТИЧЕСКОЕ РЕШЕНИЕ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ С РАЗРЫВНЫМИ КОЭФФИЦИЕНТАМИ МЕТОДОМ ТЕПЛОВЫХ ПОЛИНОМОВ

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Ключевые слова: Аналитическое решение, уравнение, теплопроводность, метод, тепловые полиномы.
Аннотация. Найдено аналитическое решение уравнения теплопроводности с разрывными коэффициентами в областях с подвижными границами, вырождающимимся в начальный момент времени методом тепловых полиномов.

1.Introduction:

The aim of this paper is to solve heat equation with the second type moving discontinuous boundary conditions that degenerate at the initial time and which is the part of research devoted to find solution of Stefan type problems by special functions necessary for modelling arc phenomena in electrical contacts.

Analytical and numerical solutions of Heat equation with moving boundaries considered in [1-6] and one of the most powerful methods is widely known a method of Heat potentials. But unfortunately this method give qualitative solution and inapplicable in cases where moving boundary degenerates at the initial time. Various methods are listed in bibliography of problems with free, moving boundaries [7] but unfortunately none of them are applicable for engineering problems particularly necessary for mathematical modelling of above mentioned physical phenomena.

We will try to answer these questions as following. In the first section some necessary properties of Integral Error Function that are used for solving heat equation with moving boundaries are represented. In the second section by the use of multinomial coefficients of Newton's polynomials solution of heat equation with second type boundary conditions developed.

1.1 Integral Error Functions

Heat equations are solved by the help of so called IEF method (Integral Error Functions or Hartree functions method) and properties of Integral Error Functions which were introduced by Hartree in 1935 and reasonably sometimes called Hartree functions.

The integral error functions determined by recurrent formulas

$$i^n erfcx = \int_x^\infty i^{n-1} erfcv dv, \quad n=1,2,\ldots \qquad i^0 erfcx \equiv erfcx = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-v^2) dv \tag{1}$$

where $$erfx = 1 - erfcx = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-v^2) dv \tag{2}$$

One can obtain from

$$i^n erfcx = \frac{2}{\sqrt{\pi}} \frac{1}{n!} \int_x^\infty (v-x)^n \exp(-v^2) dv \tag{3}$$

Expressions (1) satisfy the differential equation

$$\frac{d^2}{dx^2} i^n erfcx + 2x \frac{d}{dx} i^n erfcx - 2n i^n erfcx = 0 \tag{4}$$

and recurrent formulas

$$2n i^n erfcx = i^{n-2} erfcx - 2x i^{n-1} erfcx \tag{5}$$

Integral Error Functions are very useful for investigation of heat transfer, diffusion and other phenomena which can be described by the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

in a region $D(t > 0, 0 < x < \alpha(t))$ with free boundary $x = \alpha(t)$, since the functions

$$u_n(\pm x, t) = t^{\frac{n}{2}} i^n erfc \frac{\pm x}{2a\sqrt{t}} \tag{7}$$

suffice the equation (6) as well as their linear combination or even series

$$u(x,t) = \sum_{n=0}^{\infty} [A_n u_n(x,t) + B_n u_n(-x,t)] \tag{8}$$

For any constants A_n, B_n. We can choose these constants to satisfy the boundary conditions at $x = 0$ and $x = \alpha(t)$, if given boundary functions can be expanded into Maclaurin series with powers t or $\sqrt{t}$.

1.2 Properties of Integral Error Functions

It is possible to derive properties of Integral Error Functions.

1. Using formula for Hermite polynomials one can derive

$$u(x,t) = \sum_{n=0}^{\infty} \left[A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m} \right] \tag{9}$$

where u(x,t) is Heat polynomial which exactly satisfy Heat Equation

where $$\beta_{n,m} = \frac{1}{2^{n+m-1} m!(n-2m)!} \tag{10}$$

2. Using L'Hopital rule and representation (1), it is not difficult to show that

$$\lim_{x \to \infty} \frac{i^n erfc(-x)}{x^n} = \frac{2}{n!} \tag{11}$$

2. Problem Statement

2.1 It is required to find the solution of Heat Equation with moving (known) boundary that degenerate at the initial time

$$\frac{\partial u_1}{\partial t} = a^2 \frac{\partial^2 u_1}{\partial x^2} \qquad 0 < x < \alpha(t), \quad t > 0 \tag{12}$$

$$\frac{\partial u_2}{\partial t} = a^2 \frac{\partial^2 u_2}{\partial x^2} \qquad \alpha(t) < x < \gamma(t), \quad t > 0 \tag{13}$$

where

$$\alpha(t) = \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \cdots + \alpha_k t^k + \cdots \tag{14}$$

$$\gamma(t) = \gamma_1 t + \gamma_2 t^2 + \gamma_3 t^3 + \cdots + \gamma_k t^k + \cdots \tag{15}$$

$$I.C: \qquad u(0,0) = 0 \tag{16}$$

$$B.C: \qquad \left.\frac{\partial u_1}{\partial x}\right|_{x=0} = \psi(t) \tag{17}$$

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=\alpha(t)} = \left.\frac{\partial u_2}{\partial x}\right|_{x=\alpha(t)} \tag{18}$$

$$u_1|_{x=\gamma(t)} = u_2|_{x=\gamma(t)} \tag{19}$$

$$\left.\frac{\partial u_2}{\partial x}\right|_{x=\gamma(t)} = \varphi(t) \tag{20}$$

From property (1) section 1.1 we consider solution in the form of Heat Polynomials

$$u_1(x,t) = \sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n} x^{2n-2m}t^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n} x^{2n-2m+1}t^m\beta_{2n+1,m}\right] \tag{21}$$

$$u_2(x,t) = \sum_{n=0}^{\infty}\left[B_{2n}\sum_{m=0}^{n} x^{2n-2m}t^m\beta_{2n,m} + B_{2n+1}\sum_{m=0}^{n} x^{2n-2m+1}t^m\beta_{2n+1,m}\right] \tag{22}$$

2.2 Method of solution

$$\frac{\partial u}{\partial x} = \sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n}(2n-2m)x^{2n-2m-1}t^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)x^{2n-2m}t^m\beta_{2n+1,m}\right] \equiv$$

$$\equiv A_2 2x\beta_{2,0} +$$
$$A_4(4x^3\beta_{4,0} + 2xt\beta_{2,1}) +$$
$$A_6(6x^5\beta_{6,0} + 4x^3t\beta_{6,1} + 2xt^2\beta_{6,2}) +$$
$$A_8(8x^7\beta_{8,0} + 6x^5t\beta_{8,1} + 4x^3t^2\beta_{8,2} + 2xt^3\beta_{8,3}) +$$
$$A_{10}(10x^9\beta_{10,0} + 8x^7t\beta_{10,1} + 6x^5t^2\beta_{10,2} + 4x^3t^3\beta_{10,3} + 2xt^4\beta_{10,4}) +$$
$$A_{12}(12x^{11}\beta_{12,0} + 10x^9t\beta_{12,1} + 8x^7t^2\beta_{12,2} + 6x^5t^3\beta_{12,3} + 4x^3t^4\beta_{12,4} + 2xt^5\beta_{12,5}) +$$
$$A_{14}(14x^{13}\beta_{14,0} + 12x^{11}t\beta_{14,1} + 10x^9t^2\beta_{14,2} + 8x^7t^3\beta_{14,3} + 6x^5t^4\beta_{14,4} + 4x^3t^5\beta_{14,5} + 2xt^6\beta_{14,6}) + \cdots$$
$$\ldots + A_{2k}(2kx^{2k-1}\beta_{2k,0} + (2k-1)x^{2k-3}t\beta_{2k,1} + \cdots + 2xt^{k-1}\beta_{2k,k-1}) + \cdots +$$
$$A_1\beta_{1,0} + A_3(3x^2\beta_{3,0} + t\beta_{3,1}) +$$
$$A_5(5x^4\beta_{5,0} + 3x^2t\beta_{5,1} + t^2\beta_{5,2}) +$$
$$A_7(7x^6\beta_{7,0} + 5x^4t\beta_{7,1} + 3x^2t^2\beta_{7,2} + t^3\beta_{7,3}) +$$

$$A_9\left(9x^8\beta_{9,0}+7x^6t\beta_{9,1}+5x^4t^2\beta_{9,2}+3x^2t^3\beta_{9,3}+t^4\beta_{9,4}\right)+$$
$$A_{11}\left(11x^{10}\beta_{11,0}+9x^8t\beta_{11,1}+7x^6t^2\beta_{11,2}+5x^4t^3\beta_{11,3}+3x^2t^4\beta_{11,4}+t^5\beta_{11,5}\right)+$$
$$A_{13}\left(13x^{12}\beta_{13,0}+11x^{10}t\beta_{13,1}+9x^8t^2\beta_{13,2}+7x^6t^3\beta_{13,3}+5x^4t^4\beta_{13,4}+3x^2t^5\beta_{13,5}+t^6\beta_{13,6}\right)+$$
$$A_{15}\left(15x^{14}\beta_{15,0}+13x^{12}t\beta_{15,1}+11x^{10}t^2\beta_{15,2}+9x^8t^3\beta_{15,3}+7x^6t^4\beta_{15,4}+5x^4t^5\beta_{15,5}+3x^2t^6\beta_{15,6}+t^7\beta_{15,7}\right)+\cdots$$
$$\ldots+A_{2k+1}\left((2k+1)x^{2k}\beta_{2k+1,0}+(2k-1)x^{2k-2}t\beta_{2k+1,1}+\cdots+t^k\beta_{2k+1,k}\right)+\cdots+ \quad (23)$$

Taking k times derivatives from both sides of expression (23) we get A_{2n+1} coefficient as following

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=A_1\beta_{1,0}+A_3t\beta_{3,1}+A_5t^2\beta_{5,2}+\cdots+A_{2k+1}t^k\beta_{2k+1,k}+\cdots=\sum_{n=0}^{\infty}A_{2n+1}t^n\beta_{2n+1,n}\equiv\varphi(t) \quad (24)$$

yields

$$\sum_{n=0}^{\infty}A_{2n+1}t^n\beta_{2n+1,n}=\sum_{n=0}^{\infty}\frac{\psi^{(n)}(0)}{n!}t^n \quad (25)$$

$$A_{2n+1}=\frac{\psi^{(n)}(0)}{n!\beta_{2n+1,n}}\equiv\psi^{(n)}(0)2^{3n} \quad (26)$$

To find the remaining unknown coefficients A_{2n} we use multinomial coefficients of Newton's Polynomials.

2.3 Newton's Polynomials

$$(x_1+x_2+\cdots+x_{k+1})^n=\sum_{s_1+s_2+\cdots+s_{k+1}=n}\binom{n}{s_1,s_2,\ldots,s_{k+1}}\prod_{1\le t\le k+1}x_t^{s} \quad (27)$$

$where \binom{n}{s_1,s_2,\ldots,s_{k+1}}=\frac{n!}{s!s_2!\ldots s_{k+1}!}$ is a *multinomial coefficient*

In our case where $x_1+x_2+\cdots+x_{k+1}=\alpha(t)\equiv\sum_{n=0}^{k}\alpha_{n+1}t^{n+1}$ (27')

We have $(\alpha_1t+\alpha_2t^2+\cdots+\alpha_{k+1}t^{k+1})^n=$

$$=\sum_{s_1+s_2+\cdots+s_{k+1}=n}\binom{n}{s_1,s_2,\ldots,s_{k+1}}\alpha_1^{s_1}\alpha_2^{s_2}\ldots\alpha_{k+1}^{s_{k+1}}t^{s_1+2s_2+\cdots+(k+1)s_{k+1}} \quad (28)$$

Where

$$\binom{n}{s_1,s_2,\ldots,s_{k+1}}\alpha_1^{s_1}\alpha_2^{s_2}\ldots\alpha_{k+1}^{s_{k+1}}t^{s_1+2s_2+\cdots+(k+1)s_{k+1}} \quad (29)$$

is *a multinomial coefficient* in our case.

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=\alpha(t)}=$$
$$=\sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n}(2n-2m)x^{2n-2m-1}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)x^{2n-2m}t^m\beta_{2n+1,m}\right]\equiv$$
$$\equiv\sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n}(2n-2m)\left(\alpha(t)\right)^{2n-2m-1}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)\left(\alpha(t)\right)^{2n-2m}t^m\beta_{2n+1,m}\right] \quad (30)$$

We substitute (15) into (20) and get

$$\sum_{n=0}^{\infty}\left[B_{2n}\sum_{m=0}^{n}(2n-2m)\left(\alpha(t)\right)^{2n-2m-1}t^m\beta_{2n,m}+B_{2n+1}\sum_{m=0}^{n}(2n-2m+1)\left(\alpha(t)\right)^{2n-2m}t^m\beta_{2n+1,m}\right]=$$
$$=\sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n}(2n-2m)\sum_{s_1+s_2+\cdots+s_{k+1}=2n-2m-1}\binom{2n-2m-1}{s_1,s_2,\ldots,s_{k+1}}\alpha_1^{s_1}\alpha_2^{s_2}\ldots\alpha_{k+1}^{s_{k+1}}t^{s_1+2s_2+\cdots+(k+1)s_{k+1}+m}\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)\sum_{p_1+p_2+\cdots+p_{k+1}=2n-2m}\binom{2n-2m}{p_1,p_2,\ldots,p_{k+1}}\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_{k+1}^{p_{k+1}}t^{p_1+2p_2+\cdots+(k+1)p_{k+1}+m}\beta_{2n+1,m}\right]$$

In the same manner fulfilling above procedure with expression (20) we can easily derive recurrent formula for B_{2n}. Since $\varphi(t)$ is analytic function and can be expanded into Maclaurin we can find B_{2n} by taking both sides of expression (20) 2k and 2k+1 times derivatives and after that equate coefficients of both sides.

$$\varphi^{(2k)}(0)=\left(\sum_{n=1}^{k}B_{2n}\sum_{m=0}^{n-1}c_{2n,m}[2k](2n-2m)(2k)!\,\beta_{2n,m}\right)+\left(\sum_{n=k+1}^{2k}B_{2n}\sum_{m=0}^{2k-n}c_{2n,m+2(n-k)-1}[2k]2(2k-n-m+1)(2k)!\,\beta_{2n,m+2(n-k)-1}\right)+\left(\sum_{n=1}^{k}B_{2n+1}\sum_{m=0}^{n-1}c_{2n+1,m}[2k](2n+1-2m)(2k)!\,\beta_{2n+1,m}\right)+\left(\sum_{n=k+1}^{2k-1}B_{2n+1}\sum_{m=0}^{2k-n-1}c_{2n+1,m+2(n-k)}[2k](2(2k-n-m)+1)(2k)!\,\beta_{2n+1,m+2(n-k)}\right)+(2k)!\,B_{4k+1}c_{4k+1,2k}[2k]\beta_{4k+1,2k} \quad (31)$$

$$\varphi^{(2k+1)}(0)=\left(\sum_{n=1}^{k+1}B_{2n}\sum_{m=0}^{n-1}c_{2n,m}[2k+1](2n-2m)(2k+1)!\,\beta_{2n,m}\right)+\left(\sum_{n=k+2}^{2k+1}B_{2n}\sum_{m=0}^{2k-n+1}c_{2n,m+2(n-k-1)}[2k+1]2(2k-n-m+2)(2k+1)!\,\beta_{2n,m+2(n-k-1)}\right)+\left(\sum_{n=1}^{k}B_{2n+1}\sum_{m=0}^{n-1}c_{2n+1,m}[2k+1](2n+1-2m)(2k+1)!\,\beta_{2n+1,m}\right)+\left(\sum_{n=k+1}^{2k}B_{2n+1}\sum_{m=0}^{2k-n}c_{2n+1,m+2(n-k)-1}[2k+1](2(2k-n-m)+3)(2k+1)!\,\beta_{2n+1,m+2(n-k)-1}\right)+B_{4k+3}c_{4k+3,2k+1}[2k+1](2k+1)!\,\beta_{4k+3,2k+1} \quad (32)$$

Where $s_1+2s_2+\dots+(k+1)s_{k+1}+m=2k,\ \ p_1+2p_2+\dots+(k+1)p_{k+1}+m=2k$ for even derivatives and $s_1+2s_2+\dots+(k+1)s_{k+1}+m=2k+1,\ \ p_1+2p_2+\dots+(k+1)p_{k+1}+m=2k+1$ for odd derivatives.

substituting (26) into (18) we get

$$\sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n}(2n-2m)x^{2n-2m-1}t^m\beta_{2n,m}+\psi^{(n)}(0)2^{3n}\sum_{m=0}^{n}(2n-2m+1)x^{2n-2m}t^m\beta_{2n+1,m}\right]\Big|_{x=\alpha(t)}=\sum_{n=0}^{\infty}\left[B_{2n}\sum_{m=0}^{n}(2n-2m)x^{2n-2m-1}t^m\beta_{2n,m}+B_{2n+1}\sum_{m=0}^{n}(2n-2m+1)x^{2n-2m}t^m\beta_{2n+1,m}\right]\Big|_{x=\alpha(t)} \quad (33)$$

Yields

$$B_{4k+1}=\varphi^{(2k)}(0)/(2k)!\,c_{4k+1,2k}[2k]\beta_{4k+1,2k}-\Big\{\left(\sum_{n=1}^{k}B_{2n}\sum_{m=0}^{n-1}c_{2n,m}[2k](2n-2m)\beta_{2n,m}\right)+\left(\sum_{n=k+1}^{2k}B_{2n}\sum_{m=0}^{2k-n}c_{2n,m+2(n-k)-1}[2k]2(2k-n-m+1)\beta_{2n,m+2(n-k)-1}\right)+\left(\sum_{n=1}^{k}B_{2n+1}\sum_{m=0}^{n-1}c_{2n+1,m}[2k](2n+1-2m)\beta_{2n+1,m}\right)+\left(\sum_{n=k+1}^{2k-1}B_{2n+1}\sum_{m=0}^{2k-n-1}c_{2n+1,m+2(n-k)}[2k](2(2k-n-m)+1)\beta_{2n+1,m+2(n-k)}\right)\Big\}/c_{4k+1,2k}[2k]\beta_{4k+1,2k} \quad (34)$$

$$B_{4k+3}=\varphi^{(2k+1)}(0)/c_{4k+3,2k+1}[2k+1]\beta_{4k+3,2k+1}-\left\{\left(\sum_{n=1}^{k+1}B_{2n}\sum_{m=0}^{n-1}c_{2n,m}[2k+1](2n-2m)\beta_{2n,m}\right)+\left(\sum_{n=k+2}^{2k+1}B_{2n}\sum_{m=0}^{2k-n+1}c_{2n,m+2(n-k-1)}[2k+1]2(2k-n-m+2)\beta_{2n,m+2(n-k-1)}\right)+\left(\sum_{n=1}^{k}B_{2n+1}\sum_{m=0}^{n-1}c_{2n+1,m}[2k+1](2n+1-2m)\beta_{2n+1,m}\right)+\left(\sum_{n=k+1}^{2k}B_{2n+1}\sum_{m=0}^{2k-n}c_{2n+1,m+2(n-k)-1}[2k+1](2(2k-n-m)+3)\beta_{2n+1,m+2(n-k)-1}\right)\right\}/\,c_{4k+3,2k+1}[2k+1]\beta_{4k+3,2k+1} \quad (35)$$

for k=0,1,2,…

If we substitute coefficients (34),(35) into (33) coefficients A_{2n} can be expressed in terms of unknown even coefficients B_{2n}

By using same principle. We take both sides of expression (33) 2k and 2k+1 times derivatives obtaining recurrent formula for A_{2n} coefficients expressed in terms of B_{2n}. In the same manner using formulas for multinomial coefficients and by substituting instead of A_{2n} recurrent formula expressed in terms of B_{2n} we can find B_{2n} from (19).

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АЙЫРЫЛЫМДЫҚ КОФФИЦИЕНТТЕРІ БАР ЖЫЛУ ӨТКІЗГІШТІК ТЕҢДЕУІНІҢ ЖЫЛУ ПОЛИНОМДАРЫ АРҚЫЛЫ АНАЛИТИКАЛЫҚ ШЕШІМІ

Аннотация

Бастапқы уақытта құлдырайтын, жылжымалы шекарасы бар аймақтарда айырылымдық коэффициенттері бар жылу өткізгіштік теңдеудің жылу полиномдар арқылы аналитикалық шешімі табылған.

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