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BOUNDARY VALUE PROBLEM FOR ELASTIC HALF-SPACE BY SUBSONIC VELOCITIES OF SURFACE TRANSPORT LOADS MOVING

Abstract. The first boundary value problem of the theory of elasticity for an elastic half-space at the movement on its surface of subsonic trans loads is considered. The speed of motion is less or more than the speed of distribution of elastic Rayleigh waves. On the basis of the generalized Fourier's transformation the fundamental solution of a task is constructed which describes dynamics of the massif at the movement of the concentrated force on and along its surface. Based on this, an analytical solution is constructed for arbitrary transport loads distributed over the surface, moving with the pre- Rayleigh and super-Rayleigh velocities. It is shown that when the Rayleigh wave velocity is exceeded, the transport loads generate surface Rayleigh waves.

The task is a model for research of the intense deformed condition of the pedigree massif in the vicinity of road constructions at moving transport.

Keywords: boundary value problem, an elastic half-space, trans loading, subsonic speed, Rayleigh wave, the stress-strain state.

Trans loads are very widespread in practice. As those we understand the moving loads which form doesn't change over time, but their position are changing in the environment. Dynamic deformation processes, which arise in the ground under their influence, expand with different speeds, characterizing elastic properties of the medium. In isotropic elastic medium there are two sound speeds of expansion of *dilatation waves* (c_1) and *shift* (c_2) *waves* ($c_1 > c_2$). The relation of speed of trans load to the sound velocities significantly influences to the stresses and deformations in the elastic medium. We consider here the subsonic case, when speeds of loads are less then shift waves speed. This case is characteristic for trans problems as the speed of the movement of the most modern vehicles is many less then the speeds of elastic waves propagation. From trans loads we especially distinguish stationary ones which move in the fixed direction with a constant speed (*transport loads*). This class of loads allows to investigate diffraction processes in isotropic elastic medium in analytical form.

In papers [1-3] the fundamental and generalized solutions of the Lame's equations are constructed and investigated which describe the movement of elastic medium at the action of concentrated on an axis and distributed loading in all range of speeds (subsonic, sound, transonic and supersonic ones). On this basis in [4-7] the method of boundary integral equations has been developed for solving the transport BVP in elastic medium with cylindrical boundaries. This class of problems is very important for applications in the field of dynamics of underground constructions, trans tunnels and excavations of deep laying.

However there is a class of model trans tasks (for example, road problems) when loadings move on the surface of a half-space. It is known that there is also sound speed in an elastic half-space with which superficial Rayleigh waves are propagating. The Rayleigh's speed is less, but is very close to the speed of shift waves [10,11]. Rayleigh's waves don't create tensions on half-space border, but significantly influence on the tensions and deformations of the massif near a free surface.

For the first time such task was considered and solved for a subsonic pre-Ryleigh case by flat deformation in work [8]. Here the analytical solution of this task in three-dimensional statement is constructed also in a subsonic case, when the speed of subsonic trans load is less or more than the Rayleigh's speed.

1. **The statement of transport boundary value problem for elastic half-space**

Elastic isotropic medium, with Lame's parameters λ,μ and the density ρ, occupies half-plane $x_1>0$, ***n*** (x)=(-1,0, 0) is unit vector of the external normal to its boundary $D=\{x\in R^3 : x_1=0\}$. Constants c_1 and c_2 are the velocities of elastic waves propagation [11]:

$$c_1=\sqrt{\frac{\lambda+2\mu}{\rho}},\quad c_2=\sqrt{\frac{\mu}{\rho}},\quad c_2<c_1\ (\textit{sonic speeds}):$$

Boundary transport load $\mathbf{P}(x,t)$ are moving with constant subsonic speed ($c<c_2<c_1$) along the axis X_3: $\mathbf{P}(x,t)=\mu p_j(x_2,x_3+ct)e_j$. Components of stress tensor σ_{ij} are connected with medium displacements $u(x,t)$ by Hook's law:

$$\sigma_{ij}=\lambda\,\mathrm{div}u\,\delta_{ij}+\mu(u_{i,j}+u_{j,i})\ .$$

For dynamics problems this law better to write in the unitless form:

Hook's law:

$$\frac{\sigma_{ij}}{\mu}=\left(\frac{c_1^2}{c_2^2}-2\right)\mathrm{div}\,u\,\delta_{ij}+(u_{i,j}+u_{j,i}) \tag{1}$$

Here and everywhere further on the identical indexes the tensor convolution have been made. Private derivatives on the corresponding coordinate are designated by the index after comma: $u_{i,j}=\frac{\partial u_i}{\partial x_j}$; $\delta_{ij}=\delta_i^j$ is Kronecker symbol.

The stationary movement has been considered that allows to pass into mobile coordinates system which are connected with transport load. Further we use designations: $x=(x_1,x_2)$, $z=x_3+ct$.

It's supposed that components of the load allow the Fourier's transformation, i.e. they are representable in the form of Fourier's integrals:

$$\begin{gathered}P_j(x_2,z)=\sigma_{j1}(0,x_2,z)=\frac{\mu}{(2\pi)^2}\int_{R^2}\overline{p}_n(\eta,\varsigma)\exp(-i(x_2\eta+\zeta z))d\eta d\varsigma\\ \overline{p}_n(\eta,\varsigma)=\int_{R^2}p_n(x_2,z)\exp(i(x_2\eta+z\varsigma))dx_2dz\end{gathered} \tag{2}$$

The Lame's equations for displacements of elastic half-space in mobile coordinates system have the form [1]:

$$\left(\left(M_1^{-2}-M_2^{-2}\right)\frac{\partial^2}{\partial x_i\partial x_j}+\left(M_2^{-2}\Delta-(\partial_z)^2\right)\delta_i^j\right)u_j=0 \tag{3}$$

This operator we denote $L_{ij}(\partial_1,\partial_2,\partial_z)$. Here two Mach's numbers are introduced:

$$M_1=c/c_1,\quad M_2=c/c_2,$$

which characterize the velocity of transport load in relation to the sound speeds of elastic waves.

Eqs. (3) were studied in [2,3]. There are three cases: subsonic $(c<c_2)$, transonic$(c_2<c<c_1)$, supersonic $(c>c_1)$ and two sonic cases $(c=c_2, c=c_1)$. In the first case $(M_1<1,\ M_2<1)$ the system (3) is elliptic, in the second one $(M_1<1,\ M_2>1)$ it has have the mixed elliptic-hyperbolic type. In supersonic case $(M_1 > 1,\ M_2 > 1)$ this system is strong hyperbolic. By sonic speeds it is mixed parabolic-elliptic if $M_1 < 1,\ M_2 = 1$, and it's hyperbolic-parabolic if $M_1 = 1,\ M_2 > 1$.

By sonic and supersonic velocities the shock waves can appear in elastic medium. There are the next conditions on the jumps on their fronts F:

$$\left[u_j\right]_F = 0 \quad \Rightarrow \quad h_z\left[u_{i,j}\right]_F = h_j\left[u_{i,z}\right]_F; \qquad h_j\left[\sigma_{ij}\right]_F = \rho c^2 h_z\left[u_{i,z}\right]_F, \quad i,j = 1,2,3. \tag{4}$$

Here $h(x_1, x_2, z) = (h_1, h_2, h_3 \square h_z)$ is wave vector, $\|h\| = 1$. It's perpendicular to a front F in the direction of wave propagation. The continuity of elastic medium gives the first condition. The second condition is continuity of tangent derivatives at the front of a wave; it is consequence from the first one. The third formula is law of momentum conservation on waves fronts.

Here we consider the subsonic case. It's required to find the solution of the BVP which must to satisfy

the attenuation condition on infinity:

$u \to 0$ by $x_1 \to +\infty$ or $z \to \pm\infty$. (5)

Also we'll enter later some additional *radiation conditions* by construction of BVP solution.

1. Green tensor of transport BVP

To solve the problem, we construct the Green's tensor Π_j^k of the boundary value problem in a moving coordinates system. For its determination we have the following boundary value problem.

To find the tensor solution of homogeneous motion equations :

$$\left(\left(M_1^{-2} - M_2^{-2}\right)\frac{\partial^2}{\partial x_i \partial x_j} + \left(M_2^{-2}\Delta - \frac{\partial^2}{\partial z^2}\right)\delta_i^j\right)\Pi_j^k = 0, \quad i,j,k = 1,2,3, \tag{6}$$

in the region $x_1 > 0$, which must to satisfy the attenuation condition at infinity:

$$\Pi_j^k(x,z) \to 0 \ \text{for} \ \|(x,z)\| \to 0 . \tag{7}$$

Corresponding stress tensor Σ_{jk}^m, which are calculated by use Hooh's law (2), has the form:

$$\Sigma_{jk}^m = \alpha\Pi_l^m{}_{,l}\,\delta_{jk} + (\Pi_j^m{}_{,k} + \Pi_k^m{}_{,j}) = S_{jk}^l(\partial_1, \partial_2, \partial_z)\Pi_l^m(x_1, x_2, z), \qquad S_{jk}^l = \alpha\delta_{jk}\partial_l + (\delta_{jl}\partial_k + \delta_{lk}\partial_j) \tag{8}$$

T h e o r e m. *The solution of the given boundary-value problem has the form of the following convolution on the boundary of a half-space*

$$u_j(x_1, x_2, z) = \int_{R^q}^{\infty} \Pi_j^n(x_1, x_2 - y_2, z - y_3)\,p_n(y_2, y_3)\,dy_2 dy_3, \quad j = 1,2,3. \tag{9}$$

and must satisfy to the following singular conditions on the free surface : *for* x_1=0

$$\Sigma_{i1}^m = \alpha\Pi_k^m{}_{,k}\,\delta_{i1} + (\Pi_i^m{}_{,1} + \Pi_1^m{}_{,i}) = \delta_i^m\delta(x_2)\delta(z), \quad i,m,k = 1,2,3. \tag{10}$$

where $\delta(x_j)$ *is generalized Dirac function,* $\alpha = \dfrac{\lambda}{\mu} = \left(\dfrac{c_1^2}{c_2^2} - 2\right) = \left(\dfrac{M_2^2}{M_1^2} - 2\right)$.

P r o o f. Indeed, by virtue of (1), (10) and the convolution properties we have on the boundary of the half-space:

$$\int_{R^q}^{\infty} \Sigma_{j1}^{m}(0, x_2 - y_2, z - y_3) p_m(y_2, y_3)\, dy_2 dy_3 = \delta_j^m \delta(x_2)\delta(z) * p_m(x_2, z) = p_j(x_2, z).$$

Here on the right there is a functional convolution along the half-space boundary and tensor convolution by the index m. The displacements (9) satisfy to the homogeneous transport Lame equations (3) in the half-space:

$$L_i^j(\partial_1, \partial_2, \partial_z) u_j = \int_{R^q}^{\infty} p_n(y_2, y_3) L_i^j(\partial_1, \partial_2, \partial_z) \Pi_j^n(x_1, x_2 - y_2, z - y_3) dy_2 dy_3 = 0$$

in view of (6) and the invariance of these equations with respect to the shift at the boundary of the half-space

This tensor $\Pi(x, z)$ gives possibility to use the formula (9) for determination of displacements in a half-space for any loading on its surface. The stresses at any point of the elastic half-space on the area with the normal n are determined by the formula

$$S(x_1, x_2, z, n) = \sigma_{jk}(x_1, x_2, z) n_j e_k = \\ = \mu e_k n_j \int_{R^q}^{\infty} \Sigma_{kj}^{l}(x_1, x_2 - y_2, z - y_3) p_l(y_2, y_3)\, dy_2 dy_3. \qquad (11)$$

Thus, the definition of the fundamental displacement tensor determines the solution of the problem.

We construct the tensor $\Pi(x, z)$ using the scalar and vector elastic Lame potentials.

2. Statement of the transport problem for Lame's potentials

The displacements of the elastic medium can be represented in terms of the scalar and vector Lame's potentials [1,11]:

$$u = grad\,\varphi + rot\,\psi \qquad (12)$$

Since the three components of the displacements are determined through four potential components, the vector potential is usually associated with a Gaussian or Lorentz gauge. Here it is convenient to use the representation:

$$\psi = \psi_1 e_3 + rot(\psi_2 e_3),$$

which uniquely links the three components of displacements with three potentials. If the displacements satisfy to the homogeneous Lame equations, then the potentials satisfy the d'Alembert's wave equation with the corresponding velocity:

$$c_1^2 \Delta\varphi - \frac{\partial^2 \varphi}{\partial t^2} = 0, \\ c_2^2 \Delta\psi_k - \frac{\partial^2 \psi_k}{\partial t^2} = 0, \quad k = 1,2 \qquad (12)$$

where Δ is Laplace operator. In the moving coordinate system these equations are transformed to the form:

$$\Delta\varphi - M_1^2 \frac{\partial^2 \varphi}{\partial z^2} = 0, \\ \Delta\psi_k - M_2^2 \frac{\partial^2 \psi_k}{\partial z^2} = 0, \quad k = 1,2 \qquad (13)$$

To construct the tensor Π^i_j, we use similar potentials. Namely, we represent it in the form

$$\Pi^m_k(x_1,x_2,z)=D_{kn}(\partial_1,\partial_2,\partial_z)\Phi^m_n=\partial_k\Phi^m_1+e_{ki3}\partial_i\Phi^m_2+e_{kjl}e_{li3}\partial_j\partial_i\Phi^m_3$$

$$D_{k1}(\partial_1,\partial_2,\partial_z)=\partial_k$$
$$D_{k2}(\partial_1,\partial_2,\partial_z)=e_{ki3}\partial_i \qquad (14)$$
$$D_{k3}(\partial_1,\partial_2,\partial_z)=e_{kjl}e_{li3}\partial_i\partial_j$$

Here i,j,k,l,m=1,2,3, e_{ijk} is Levi-Civita pseudotensor. The first potential describes the gradient component of the displacements field, and the other two potentials describe the rotor (solenoidal) components . The potentials satisfy to the *transport wave equations*:

$$\Delta\Phi^m_j-M_j^2\frac{\partial^2\Phi^m_j}{\partial z^2}=0,\quad j=1,2,3. \qquad (15)$$

We name them *fundamental potentials.* To calculate them we use boundary conditions (9) :

by $x_1=0$

$$\alpha\,\Pi^m_{k,k}\,\delta_{i1}+(\Pi^m_{i,1}+\Pi^m_{1,i})=\delta^m_i\delta(x_2)\delta(z)$$

where

$$\Pi^m_{k,k}=\Delta\Phi^m_1+e_{ki3}\partial_k\partial_i\Phi^m_2+e_{kjl}e_{li3}\partial_k\partial_i\partial_j\Phi^m_3\,,$$
$$\Pi^m_{i,1}=\partial_i\partial_1\Phi^m_1+e_{ik3}\partial_k\partial_1\Phi^m_2+e_{ijl}e_{lk3}\partial_k\partial_j\partial_1\Phi^m_3\,,$$
$$\Pi^m_{1,i}=\partial_i\partial_1\Phi^m_1+e_{1k3}\partial_k\partial_i\Phi^m_2+e_{1jl}e_{lk3}\partial_k\partial_j\partial_i\Phi^m_3\,,$$

We can to write it in the form:

$$\mathrm{B}_{in}(\partial_1,\partial_2,\partial_z)\Phi^m_n=\delta^m_i\delta(x_2)\delta(z),\quad n,m=1,2,3, \qquad (16)$$

where

$$\mathrm{B}_{in}\Phi^m_n=\left[2\partial_i\partial_1\Phi^m_1+\partial_k\left\{(e_{ik3}\partial_1+e_{1k3}\partial_i)\Phi^m_2+\partial_j\left(e_{ijl}e_{lk3}\partial_1+e_{1jl}e_{lk3}\partial_i\right)\Phi^m_3\right\}\right]+$$
$$+\alpha\left[\Delta\Phi^m_1+e_{kj3}\partial_k\partial_j\Phi^m_2+e_{kjl}e_{ls3}\partial_k\partial_s\partial_j\Phi^m_3\right]\delta_{i1}\Rightarrow$$
$$\mathrm{B}_{in}(\partial_1,\partial_2,\partial_z)\Phi^m_n=(\alpha\delta_{i1}\Delta+2\partial_1\partial_i)\Phi^m_1+\partial_k\left(\alpha\delta_{i1}e_{kj3}\partial_j+e_{ik3}\partial_1+e_{1k3}\partial_i\right)\Phi^m_2+$$
$$+\partial_k\partial_j\left\{\alpha\delta_{i1}e_{kjl}e_{ls3}\partial_s+\left(e_{ijl}e_{lk3}\partial_1+e_{1jl}e_{lk3}\partial_i\right)\right\}\Phi^m_3=$$
$$=\left(\alpha M_1^2\delta_{i1}\partial_z\partial_z+2\partial_1\partial_i\right)\Phi^m_1+\partial_k\left(\alpha\delta_{i1}e_{kj3}\partial_j+e_{ik3}\partial_1+e_{1k3}\partial_i\right)\Phi^m_2+$$
$$+\partial_k\partial_j\left\{\alpha\delta_{i1}e_{kjl}e_{ls3}\partial_s+\left(e_{ijl}e_{lk3}\partial_1+e_{1jl}e_{lk3}\partial_i\right)\right\}\Phi^m_3$$

This implies

$$\mathrm{B}_{i1}(\partial_1,\partial_2,\partial_z)=\left(\alpha M_1^2\delta_{i1}\partial_z\partial_z+2\partial_1\partial_i\right),$$
$$\mathrm{B}_{i2}(\partial_1,\partial_2,\partial_z)=\partial_k\left(\alpha\delta_{i1}e_{kj3}\partial_j+e_{ik3}\partial_1+e_{1k3}\partial_i\right)$$
$$\mathrm{B}_{i3}(\partial_1,\partial_2,\partial_z)=\partial_k\partial_j\left\{\alpha\delta_{i1}e_{kjl}e_{ls3}\partial_s+\left(e_{ijl}e_{lk3}\partial_1+e_{1jl}e_{lk3}\partial_i\right)\right\}$$

Using the properties of the permutation of the indices of the Levi-Civita tensor and the formula for its convolution:

$$e_{lij}e_{lkm}=\delta_{ik}\delta_{jm}-\delta_{im}\delta_{kj}\,,$$

these operators can be greatly simplified:

$$\mathrm{B}_{11}(\partial_1,\partial_2,\partial_z)=\left(\alpha M_1^2\partial_z^2+2\partial_1^2\right),$$

$$\mathrm{B}_{21}(\partial_1,\partial_2,\partial_z)=2\partial_1\partial_2,\quad \mathrm{B}_{31}(\partial_1,\partial_2,\partial_z)=2\partial_1\partial_3,$$

$$\mathrm{B}_{12}(\partial_1,\partial_2,\partial_z)=\partial_k\left(\alpha e_{kj3}\partial_j+e_{1k3}\partial_1+e_{1k3}\partial_1\right)=\left(\alpha e_{kj3}\partial_k\partial_j+2\partial_1\partial_2\right)=$$

$$=\alpha\left(e_{123}\partial_1\partial_2+e_{213}\partial_2\partial_1\right)+2\partial_1\partial_2=2\partial_1\partial_2\ ,$$

$$\mathrm{B}_{22}(\partial_1,\partial_2,\partial_z)=\partial_k\left(\alpha\delta_{21}e_{kj3}\partial_j+e_{2k3}\partial_1+e_{1k3}\partial_2\right)=$$

$$=\left(e_{213}\partial_1\partial_1+e_{123}\partial_2\partial_2\right)=\partial_2\partial_2-\partial_1\partial_1,$$

$$\mathrm{B}_{32}(\partial_1,\partial_2,\partial_z)=\partial_k\left(e_{3k3}\partial_1+e_{1k3}\partial_3\right)=\partial_2\partial_3,$$

$$\mathrm{B}_{13}(\partial_1,\partial_2,\partial_z)=\partial_k\partial_j\left\{\alpha e_{kjl}e_{lm3}\partial_m+\left(e_{1jl}e_{lk3}\partial_1+e_{1jl}e_{lk3}\partial_1\right)\right\}=$$

$$=\alpha\left(\delta_{km}\delta_{j3}-\delta_{kj}\delta_{m3}\right)\partial_k\partial_j\partial_m+\left(\delta_{1k}\delta_{j3}-\delta_{13}\delta_{jk}\right)\partial_1\partial_k\partial_j+$$

$$+\left(\delta_{1k}\delta_{j3}-\delta_{13}\delta_{jk}\right)\partial_1\partial_k\partial_j=\alpha\left(\partial_3\partial_m\partial_m-\partial_3\partial_j\partial_j\right)+2\partial_1\partial_1\partial_3=2\partial_1\partial_1\partial_3,$$

$$\mathrm{B}_{23}(\partial_1,\partial_2,\partial_z)=e_{2jl}e_{lk3}\partial_1\partial_k\partial_j+e_{1jl}e_{lk3}\partial_2\partial_k\partial_j=$$

$$=\left(\delta_{2k}\delta_{j3}-\delta_{23}\delta_{jk}\right)\partial_1\partial_k\partial_j+\left(\delta_{1k}\delta_{j3}-\delta_{13}\delta_{jk}\right)\partial_2\partial_k\partial_j=2\partial_1\partial_2\partial_3,$$

$$\mathrm{B}_{33}(\partial_1,\partial_2,\partial_z)=e_{2jl}e_{lk3}\partial_1\partial_k\partial_j+e_{1jl}e_{lk3}\partial_2\partial_k\partial_j=$$

$$=\left(\delta_{2k}\delta_{j3}-\delta_{23}\delta_{kj}\right)\partial_1\partial_k\partial_j+\left(\delta_{1k}\delta_{j3}-\delta_{13}\delta_{jk}\right)\partial_2\partial_k\partial_j=2\partial_1\partial_2\partial_3.$$

As a result, we get:

$$\begin{gathered}\mathrm{B}_{11}=\left(\alpha M_1^2\partial_z\partial_z+2\partial_1^2\right),\ B_{12}=2\partial_1\partial_2,\quad B_{13}=2\partial_1^2\partial_3,\\ \mathrm{B}_{21}(\partial_1,\partial_2,\partial_z)=2\partial_1\partial_2,\quad \mathrm{B}_{22}(\partial_1,\partial_2,\partial_z)=\partial_2\partial_2-\partial_1\partial_1,\quad \mathrm{B}_{23}(\partial_1,\partial_2,\partial_z)=2\partial_1\partial_2\partial_3,\\ \mathrm{B}_{31}(\partial_1,\partial_2,\partial_z)=2\partial_1\partial_3,\quad \mathrm{B}_{32}(\partial_1,\partial_2,\partial_z)=2\partial_3\partial_2,\quad \mathrm{B}_{33}(\partial_1,\partial_2,\partial_z)=2\partial_1\partial_2\partial_3.\end{gathered}\qquad(17)$$

Thus, the problem of constructing the transformants of the unknown tensors reduces to determining the Lamé potentials satisfying equations (14), the boundary conditions on the free surface (16), and the damping conditions at infinity:

$$\Phi_j^k\to 0\ \text{by}\ \left\|(x,z)\right\|\to\infty\ ,\qquad(18)$$

and certain radiation conditions, which we discuss below.

3. Determination of Fourier transforms of fundamental potentials

To construct the solution, we use the Fourier transform of the potentials with respect to x_2, z. In the space of Fourier transforms, they correspond to variables η,ζ. Their Fourier transforms are defined by the relations:

$$\overline{\Phi}^m=\int_{R^2}\Phi^m(x,z)\exp(i\eta x_2+i\zeta z)dzdx_2\ ,\ \Phi^m=\frac{1}{4\pi^2}\int_{R^2}\overline{\Phi}^m(x,\eta,\varsigma)\exp(-i\eta x_2-i\zeta z)d\varsigma d\eta$$

In the space of Fourier transforms the equations for the potentials (14) have the form:

$$\frac{d^2\overline{\Phi}_j^m}{dx_1^2}-\eta^2\overline{\Phi}_j^m-\alpha_j^2\zeta^2\overline{\Phi}_j^m=0,\quad \alpha_j=\sqrt{1-M_j^2}\ ,\ j=1,2,3.\qquad(19)$$

The expression under radical is positive, because we consider the subsonic case. The boundary conditions are transformed to the form:

$$\mathrm{B}_{ik}(\partial_1,-i\eta,-i\varsigma)\overline{\Phi}_k^m(x_1,\eta,\zeta)=\delta_i^m \text{ by } x_1=0. \tag{20}$$

Conditions for damping at infinity: for $\forall\,\eta,\zeta$

$$\overline{\Phi}_k^m(x_1,\eta,\zeta)\to 0 \text{ by } x_1\to\infty. \tag{21}$$

By these conditions the solution of Eq. (19) has the form:

$$\overline{\Phi}_j^k=\varphi_j^k(\eta,\varsigma)\exp\left(-x_1\sqrt{\eta^2+\alpha_j^2\zeta^2}\right),\ \mathrm{Re}\sqrt{\eta^2+\alpha_j^2\zeta^2}\ge 0. \tag{22}$$

Functions $\varphi_j^k(\eta,\varsigma)$ are determined from boundary conditions (19) :

$$\sum_{j=1}^{3}\mathrm{B}_{in}(-\sqrt{\eta^2+\alpha_j^2\zeta^2},-i\eta,-i\zeta)\varphi_n^m=\delta_i^m,\ k=1,2,3 \tag{23}$$

Thus, for each fixed m, we have the linear system of three equations for determination φ_k^m, from which we find

$$\varphi_j^m=\frac{\Delta_j^m(\eta,\zeta)}{\Delta(\eta,\zeta)}. \tag{24}$$

Here Δ_j^m is corresponding algebraic complement, and the denominator is equal to

$$\Delta(\eta,\zeta)=\det\{\mathrm{B}_{kj}(-\sqrt{\eta^2+\alpha_j^2\zeta^2},-i\eta,-i\zeta)\}.$$

This is a well-known Rayleigh determinant. In this case it has the form:

$$\Delta=4\nu^2\sqrt{\nu^2-M_1^2\zeta^2}\sqrt{\nu^2-M_2^2\zeta^2}-(2\nu^2-M_2^2\zeta^2)^2,\ \nu^2=\zeta^2+\eta^2. \tag{25}$$

The properties of Rayleigh determinant are known. For transport problems, it was well studied in [1]. In particular,

$$\Delta(\eta,\zeta)=0$$

$$\text{by } \eta=\eta_R^{\pm}(\varsigma)=\pm|\zeta|\sqrt{M_R^2-1}\ \Leftrightarrow\ \varsigma=\zeta_R^{\pm}(\eta)=\pm\frac{|\eta|}{\sqrt{M_R^2-1}} \tag{26}$$

where $M_R=c/c_R$, c_R is the velocity of the surface Rayleigh wave, which is subsonic ($c_R<c_2$). It can be determined from the equation:

$$4\sqrt{1-m_1^2}\sqrt{1-m_2^2}-(2-m_2^2)^2=0,\ m_j=c_R/c_j \tag{27}$$

Formulae (22), (24) formally resolve the problem in the potentials. However, in order to reconstruct the originals, it is necessary to investigate the properties of the transformants - integrand functions in (18), which essentially depend on the speed of a transport load.

6. Restoration of originals Π_k^m and Σ_{jk}^m by pre-Releygh speed c

From (14) we get

$$\overline{\Pi}_k^m=D_{kn}(\partial_1,-i\eta,-i\varsigma)\overline{\Phi}_n^m(x_1,\eta,\varsigma)=\frac{\Delta_n^m(\eta,\zeta)}{\Delta(\eta,\zeta)}D_{kn}(\partial_1,-i\eta,-i\varsigma)\exp\left(-x_1\sqrt{\eta^2+\alpha_n\varsigma^2}\right)\Rightarrow$$

$$\overline{\Pi}_k^m=\frac{\Delta_n^m(\eta,\zeta)}{\Delta(\eta,\zeta)}D_{kn}(-\sqrt{\eta^2+\alpha_n\varsigma^2},-i\eta,-i\varsigma)\exp\left(-x_1\sqrt{\eta^2+\alpha_n\varsigma^2}\right) \tag{28}$$

$$\overline{\Pi}_k^m(x_1,\eta,\zeta) = D_{kn}(-\sqrt{\eta^2+\alpha_j^2\zeta^2}, -i\eta, -i\zeta)\varphi_n^m(x_1,\eta,\zeta)\exp(-x_1\sqrt{\eta^2+\alpha_n^2\zeta^2}) \quad (28)$$

Using the inverse Fourier transform, we obtain

$$\Pi_k^m(x_1,x_2,z) = (2\pi)^{-2}\int_{R^2}\overline{\Pi}_k^m(x_1,\eta,\zeta)\exp(-i(\eta x_2+\zeta z))d\varsigma d\eta =$$

$$= (2\pi)^{-2}\int_{R^2} D_{kn}(-\sqrt{\eta^2+\alpha_j^2\zeta^2}, i\eta, i\zeta)\varphi_n^m(\eta,\zeta)\exp(-x_1\sqrt{\eta^2+\alpha_j^2\zeta^2} - i\eta x_2 - i\zeta z)d\varsigma d\eta = \quad (29)$$

$$= (2\pi)^{-2}\int_{R^2}\frac{D_{kn}(-\sqrt{\eta^2+\alpha_j^2\zeta^2}, i\eta, i\zeta)\Delta_n^m(\eta,\zeta)}{\Delta(\eta,\zeta)}\exp(-x_1\sqrt{\eta^2+\alpha_j^2\zeta^2} - i\eta x_2 - i\zeta z)d\varsigma d\eta$$

Let us calculate the fundamental stresses and their transformants. For this, we use the formulas (10), from which we obtain

$$\Sigma_{jk}^m = \lambda\Pi_{l\ ,l}^m\delta_{jk} + \mu(\Pi_{j\ ,k}^m + \Pi_{k\ ,j}^m) = S_{jk}^l(\partial_1,\partial_2,\partial_z)\Pi_l^m =$$

$$= S_{jk}^l(\partial_1,\partial_2,\partial_z)D_{ln}(\partial_1,\partial_2,\partial_z)\Phi_n^m(x_1,x_2,z) = T_{jkn}(\partial_1,\partial_2,\partial_z)\Phi_n^m(x_1,x_2,z), \quad (30)$$

$$T_{jkn} = S_{jk}^l(\partial_1,\partial_2,\partial_z)D_{ln}(\partial_1,\partial_2,\partial_z)$$

Hence we get

$$\overline{\Sigma}_{jk}^m = T_{jkn}(-\sqrt{\eta^2+\alpha_n\varsigma^2}, -i\eta, -i\varsigma)\hat{\Phi}_n^m(x_1,\eta,\varsigma) =$$

$$= T_{jkn}(-\sqrt{\eta^2+\alpha_n\varsigma^2}, -i\eta, -i\varsigma)\frac{\Delta_n^m(\eta,\zeta)}{\Delta(\eta,\zeta)}\exp\left(-x_1\sqrt{\eta^2+\alpha_n\varsigma^2}\right)$$

The original of the stress tensor in any point (x, z) is calculated by use the formula

$$\Sigma_{jk}^m(x_1,x_2,z) = (2\pi)^{-2}\int_{R^2}\overline{\Sigma}_{jk}^m(x_1,\eta,\zeta)\exp(-i(\eta x_2+\zeta z))d\varsigma d\eta. \quad (31)$$

For $c < c_R$ the determinant $\Delta(\eta,\zeta) \neq 0$ for any real ζ,η. That is, at the pre-Releygh velocities all the integrands are continuous and tend exponentially to zero when (η,ζ) tends to infinity. Therefore, the integrals exist and satisfy the damping conditions at infinity.

When $x_1 = 0$, $(x_2, z) \neq (0,0)$, the integrands in (29) and (31) are also continuous and integrable, since they are oscillating and have the order of damping not lower $O\left((\eta^2+\varsigma^2)^{-1}\right)$.

7. Determination of displacements and stresses at pre-Rayleigh speeds of transport load

To calculate the displacements of the medium for arbitrary transport load, we find the Fourier transform of the displacements. According to (9) and the convolution properties

$$\overline{u}_j(x_1,\eta,\zeta) = F_{x_2,z}[u_j(x_1,x_2,z)] = \overline{\Pi}_j^n(x_1,\eta,\varsigma)\overline{p}_n(\eta,\varsigma). \quad (32)$$

Substituting it in (28), we have

$$\overline{u}_k(x_1,\eta,\zeta) = \frac{\overline{p}_m(\eta,\varsigma)\Delta_n^m(\eta,\zeta)}{\Delta(\eta,\zeta)}D_{kn}(-\sqrt{\eta^2+\alpha_n\varsigma^2}, -i\eta, -i\varsigma)\exp\left(-x_1\sqrt{\eta^2+\alpha_n\varsigma^2}\right)$$

Returning to the original, we obtain formulas for calculating the displacements at pre-Rayleigh speeds:

$$u_k(x_1,x_2,z)=\frac{1}{4\pi^2}\iint_{R^2}\overline{u}_k(x_1,\eta,\zeta)\exp(-i(x_2\eta+z\varsigma))d\eta d\varsigma$$

To determine the stresses, we use the formula (11), which for the Fourier transforms has the type:

$$\sigma_{kj}(x_1,x_2,z)=\frac{1}{4\pi^2}\int_{R^2}\Sigma_{kj}^n(x_1,\eta,\zeta)\overline{p}_n(\eta,\varsigma)\exp(-i(x_2\eta+z\varsigma))d\eta d\zeta.$$

At pre-Rayleigh velocities in formulas (31) and (32), all the integrands are continuous and tend exponentially to zero by $x_1\to\infty$. Therefore, the integrals exist and satisfy the damping conditions at infinity. The asymptotic behavior of displacements at infinity is determined by the asymptotic of the transport load on the surface of the half-space.

6. Construction of Green tensor Π_k^m at super-Rayleigh speed ($c_R<c<c_2$)

If subsonic speed c is more then Rayleigh speed c_R then for constructing the solution we transform contour of integration in the ε-vicinity of point $\zeta_R(\eta)$ by any fixed η by way of moving along the circle of radius ε in upper half-plane of complex ζ by z>0 and in under half-plane by z<0 to get under sign of integral the waves, which tend to zero by $|z|\to\infty$. If $\varepsilon\to 0$, then, with use the theorem on residue of complex analysis, we get Green tensor in the form:

$$4\pi^2\Pi_k^m(x_1,x_2,z)=$$

$$=\int_{-\infty}^{\infty}\left\{\text{V.P.}\int_{-\infty}^{\infty}\sum_{j=1}^{3}D_{kn}(-\sqrt{\eta^2+\alpha_j^2\zeta^2},i\eta,i\zeta)\frac{\Delta_n^m(\eta,\zeta)}{\Delta(\eta,\zeta)}\exp(-x_1\sqrt{\eta^2+\alpha_j^2\zeta^2}-i\zeta z)d\varsigma\right\}e^{-i\eta x_2}d\eta- \tag{33}$$

$$-i\pi\,\text{sgn}\,z\sum_{\pm}\int_{-\infty}^{\infty}\sum_{j=1}^{3}D_{kn}\left(-|\eta|\sqrt{\frac{M_R^2-M_j^2}{M_R^2-1}},i\eta,i\zeta_R^{\pm}\right)\frac{\Delta_n^m(\eta,\zeta_R^{\pm})}{\Delta_\zeta(\eta,\zeta_R^{\pm}(\eta))}\exp\left(-x_1|\eta|\sqrt{\frac{M_R^2-M_j^2}{M_R^2-1}}\right)e^{-i(\eta x_2+z\zeta_R^{\pm}(\eta))}d\eta$$

Here to calculate the Value Principle integral we can use the formula:

$$\text{V.P.}\int_{-\infty}^{\infty}D_{kn}(-\sqrt{\eta^2+\alpha_j^2\zeta^2},i\eta,i\zeta)\frac{\Delta_n^m(\eta,\zeta)}{\Delta(\eta,\zeta)}\exp(-x_1\sqrt{\eta^2+\alpha_j^2\zeta^2}-i\zeta z)d\varsigma=$$

$$=\int_0^{\infty}\left(\Upsilon_{kn}^m(x_1,z,\eta,\varsigma)+\Upsilon(x_1,z,\eta,-\varsigma)\right)\exp(-x_1\sqrt{\eta^2+\alpha_j^2\zeta^2})d\varsigma,$$

$$\Upsilon_{kn}^m(x_1,z,\eta,\varsigma)=D_{kn}(-\sqrt{\eta^2+\alpha_j^2\zeta^2},i\eta,i\zeta)\frac{\Delta_n^m(\eta,\zeta)e^{-i\zeta z}}{\Delta(\eta,\zeta)}$$

The last integral doesn't have singularities in Rayleigh's points and can be calculated numerically.

The second summand in formula (33) describes the surface Rayleigh waves, which are generated by transport load when $c_R<c<c_2$.

By $c=c_R$ the stationary solution of this problem doesn't exist.

Conclusion. Here presented solutions of boundary value problems are very useful for applications when assessing the impact of road trans on the environment. It allows to determine the stress-strain state of the rock massif, depending on its elastic properties, the type of the acting load and the speed of the vehicle. This is especially true now with the development of high-speed road and rail trans, the speed of which can have a devastating impact on the surrounding areas. The obtained solutions allow us to determine the range of possible speeds of movement, taking into account the strength properties of the rock massif and the road surface, which makes it possible to ensure the safety and reliability of operation of modern vehicles.

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КРАЕВАЯ ЗАДАЧА ДЛЯ УПРУГОГО ПОЛУПРОСТРАНСТВА ПРИ ДОЗВУКОВЫХ СКОРОСТЯХ ДВИЖЕНИЯ ПОВЕРХНОСТНОЙ НАГРУЗКИ

Аннотация. Рассматривается первая краевая задача теории упругости для упругого полупространства при движении по его поверхности транспортной нагрузки со скоростью, меньшей, чем скорость распространения упругих волн. На основе обобщенного преобразования Фурье построен тензор Грина - фундаментальное решение задачи, описывающее динамику массива при движении сосредоточенной силы по его поверхности. На его основе построено аналитическое решение для произвольных распределенных по поверхности транспортных нагрузок, движущихся с дорелеевской и сверхрелеевской скоростью. Показано, что при превышении скорости волны Релея транспортные нагрузки генерируют поверхностные релеевские волны.

Задача является модельной для исследования напряженно-деформированного состояния породного массива в окрестности дорожных сооружений при движущемся транспорте.

Ключевые слова: упругое полупространство, транспортная нагрузка, дозвуковая скорость, волны Релея, напряженно-деформированное состояние.