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ON THE SECOND CHARACTERISTIC NUMBER OF THE NEWTON POTENTIAL

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Abstract. In this note we prove that the minimum of the second characteristic number of the Newton potential among bounded open sets of R^d with given volume is achieved by the union of two identical balls. The Newton potential can be related to a nonlocal boundary value problem for the Laplacian, so we obtain results on the second eigenvalue of the nonlocal Laplacian as well.

1 Introduction. LetΩ be a bounded open domain in $R^d, d \geq 3$.Consider the Newton potential operator $N: L_2(\Omega) \to L_2(\Omega)$

$$Nf := \int_{\Omega} \varepsilon_d(x-y)f(y)dy \tag{1}$$

where

$$\varepsilon_d(x-y) = \frac{1}{(d-2)\sigma_d|x-y|^{d-2}}, \qquad d \geq 3, \tag{2}$$

and$\sigma_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$is the surface area of the unit sphere in R^d.

Since ε_d is real and symmetric function N is self-adjoint operator. Therefore, all characteristic numbers are real. In addition, it is easy to check that the operator N is positive. This means all its eigenvalues are positive. The characteristic numbers N of may be enumerated in ascending order,

$$\mu_1 \leq \mu_2 \leq \cdots$$

whereμ_i is repeated in this series according to its multiplicity. We denote the corresponding eigenfunctions by $u_i, u_i, \ldots$, so that for each characteristic number μ_i there is one and only one, corresponding eigenfunction u_i,

$$u_i = \mu_i N u_i, \; i = 1,2, \ldots.$$

In a bounded domain Ω of the Euclidean space R^d, it is very well known that the solution to the Laplacian equation

$$-\Delta u(x) = f(x), \qquad x \in \Omega, \tag{3}$$

is given by the Newton potential formula

$$u(x) = \int_{\Omega} \varepsilon_d(x-y)f(y)dy, \qquad x \in \Omega, \tag{4}$$

for suitable functions f supported in Ω. An interesting question having several important applications is what boundary conditions can be put on u on the (smooth) boundary $\partial\Omega$ so that equation (3) complemented by this boundary condition would have the solution in still given by the same formula (4), with the same kernel ε_d given by (2). It turns out that the answer to this question is the integral boundary condition [4]

$$-\frac{u(x)}{2}+\int_{\partial\Omega}\frac{\partial\varepsilon_d(x-y)}{\partial n_y}u(y)dy-\int_{\partial\Omega}\varepsilon_d(x-y)\frac{\partial u(y)}{\partial n_y}dy=0\,,x\in\partial\Omega, \tag{5}$$

where $\frac{\partial}{\partial n_y}$ denotes the outer normal derivative at a point y on $\partial\Omega$. A converse question to the one above would be to determine the trace of the Newton potential (4) on the boundary surface $\partial\Omega$, and one can use the potential theory to show that it has to be given by (5).

In [4] by using the boundary condition (5) the eigenvalues and eigenfunctions of the Newton potential were explicitly calculated in the 2-disk and in the 3-ball. In general, the boundary value problem (3)-(5) has various interesting properties and applications (see, for example, Kac [2, 3] and Saito [8]). The boundary value problem (3)-(5) can also be generalized for higher degrees of the Laplacian, see [5]. In this paper we are interested in some spectral geometry questions of N.

Historically, for the first time in the scientific literature, in Rayleigh's famous book "Theory of Sound" (first published in 1877), by using some explicit computation and physical interpretations, he stated that a circle minimizes (among all domains of the same area) the first eigenvalue of the Dirichlet Laplacian. The musical interpretation of this result could be: among all drums of given area, the circular drum is the one which produces the deepest bass note. The proof of this conjecture was obtained after some decades later, simultaneously (and independently) by G.Faber and E.Krahn. Nowadays, the Rayleigh-Faber-Krahn inequality has been expanded many other operators; see [6] for further references.

In Section 2 we prove the following Rayleigh-Faber-Krahn theorem for the Newton potential N, i.e. it is proved that a ball is minimizer of the first characteristic number of the Newton potential N among all domains of given volume in R^d.

In Section 3 we are interested in minimizing the second characteristic number of N among open sets of given volume. We show that the minimizer is no longer one ball, but two! The similar result for the Dirichlet Laplacian called Krahn-Szego theorem, that is, the minimum of the second eigenvalue of the Dirichlet Laplacian among bounded open sets of R^d with given volume is achieved by the union of two identical balls. See, for example, [1] for further references.

2 Rayleigh-Faber-Krahn theorem. In this section we prove the following analogy of the Rayleigh-Faber-Krahn theorem for the Newton potential N.

Theorem 1. *A ball Ω^* is minimizer of the first characteristic number of the Newton potential N among all domains of given volume, i.e.*

$$\mu_1(\Omega^*)\le\mu_1(\Omega) \tag{6}$$

for an arbitrary bounded open domain $\Omega\subset R^d$ with $|\Omega|=|\Omega^*|$.

We will use this result later in the proof of Theorem 2.

Proof of Theorem 1. Slightly different statement of Lemma 1 is calledJentsch's theorem in [9]. However, for completeness of this note we restate and give its proof below.

Lemma 1.*The smallest characteristic number μ_1 of N is simple; the corresponding eigenfunction u_1 is positive and any other eigenfunction$u_i, i\neq 1$ is sign changing in Ω.*

Proof. The eigenfunctions of Nmay be chosen to be real as its kernel is real. First let us prove that u_1 cannot change sign in the domain Ω, that is,

$$u_1(x)u_1(y)=|u_1(x)u_1(y)|,\qquad x,y\in\Omega.$$

In fact, in the opposite case, by virtue of the continuity of the function $u_1(x)$, there would be neighborhoods $U(x_0,r)\subset\Omega$ and $U(y_0,r)\subset\Omega$ such that

$$|u_1(x)u_1(y)|>u_1(x)u_1(y), x\in U(x_0,r)\subset\Omega, y\in U(y_0,r)\subset\Omega.$$

And so, by virtue of

$$\int_\Omega\varepsilon_d(x-\xi)\varepsilon_d(\xi-y)d\xi>0. \tag{7}$$

We obtain

$$\frac{(N^2|u_1|,|u_1|)}{||u_1||^2}=\frac{1}{||u_1||^2}\int_\Omega\int_\Omega\int_\Omega\varepsilon_d(x-\xi)\varepsilon_d(\xi-y)\,d\xi|u_1(x)||u_1(y)|dxdy$$

$$>\frac{1}{||u_1||^2}\int_\Omega\int_\Omega\int_\Omega\varepsilon_d(x-\xi)\varepsilon_d(\xi-y)\,d\xi u_1(x)u_1(y)dxdy=\frac{1}{\mu_1^2}. \tag{8}$$

$\mu_1^2$is the smallest characteristic number of N^2 and u_1 is the eigenfunction corresponding to μ_1^2, i.e.

$$u_1 = \mu_1^2 N^2 u_1.$$

Therefore, by the variational principle we have

$$\frac{1}{\mu_1^2} = sup_{f\in L^2(\Omega)} \frac{(N^2 f, f)}{||f||^2}. \qquad (9)$$

This means that the strong inequality (8) contradicts the variational principle (9).

Now we shall prove that the eigenfunction$u_1(x)$ cannot become zero in Ω and therefore can be chosen positive in Ω.

In fact, in the opposite case there will be a point $x_0 \in \Omega$ such that

$$u_1(x_0) = \mu_1^2 \int_\Omega \int_\Omega \varepsilon_d(x_0 - \xi)\varepsilon_d(\xi - y)\, d\xi u_1(y) dy = 0$$

form which, by virtue of the condition (7), the contradiction follows: $u_1(y) = 0, \forall y \in \Omega$.

Since u_1 is positive it follows that μ_1 is a simple. In fact, if there were an eigenfunction $\widetilde{u_1}$ linearly independent of u_1 and corresponding to μ_1, then for all real c linear combination $u_1 + c\widetilde{u_1}$ also would be eigenfunction corresponding to μ_1 and therefore, by what has been proved, it could not become zero in Ω. As c is arbitrary, this is impossible.

Finally, we show that the other eigenfunction$u_i, i = 2,3, \ldots$, are sing changing in Ω. If $u_i \geq 0, i \neq 1$, or $u_i \leq 0, i \neq 1$ then

$$\int_\Omega u_1\, u_i \neq 0$$

as $u_1(x) > 0$ in Ω. This contradicts the orthogonality of the eigenfunction family $\{u_i\}$of N.

Lemma 1 is proved.

Let Ω be a bounded measurable set in R^d. Its symmetric rearrangement $\Omega^*$is an open ball originated at 0 with a measure equal to the measure of Ω, i.e. $|\Omega^*| = |\Omega|$. Let u be a nonnegative measurable function in Ω, in the sense that all its positive level sets have finite measure,

$$Vol(\{x | u(x) > t\}) < \infty, (\forall t > 0).$$

In the definition of the symmetric-decreasing rearrangement of u can be used the layer-cake decomposition [7], which expresses a nonnegative function u in terms of its level sets as

$$u(x) = \int_0^\infty X\{u(x) > t\} dt$$

where$X$ is the characteristic function of the corresponding domain.

Definition 1. [7]*Let u be a nonnegative measurable function in Ω. A function*

$$u^*(x) = \int_0^\infty X\{u(x) > t\}^*\, dt$$

is called a symmetric-decreasing rearrangement of a nonnegative measurable function u.

By Lemma 1 the first characteristic number μ_1 of the operator N is positive and simple; the corresponding eigenfunction u_1 can be chosen positive in Ω. Recalling Riesz' inequality [7] and the fact that $\varepsilon_d(x - y)$ is a symmetric-decreasing function, i.e. ε_d and $\varepsilon_d{}^*$ have the same formula, we obtain

$$\int_\Omega \int_\Omega u_1(y)\, \varepsilon_d(x - y) u_1(x) dy dx \leq \int_{\Omega^*} \int_{\Omega^*} u_1{}^*(y) \varepsilon_d(x - y)\, u_1{}^*(x) dy dx. \qquad (10)$$

In addition, for each nonnegative function $u \in L^2(\Omega)$ we have

$$||u||_{L^2(\Omega)} = ||u^*||_{L^2(\Omega^*)}. \qquad (11)$$

Therefore, from (10), (11) and the variational principle for $\mu_1(\Omega^*)$, we get

$$\mu_1(\Omega) = \frac{\int_\Omega |u_1(x)|^2 dx}{\int_\Omega \int_\Omega u_1(y)\theta(x-y) u_1(x) dy dx} \geq \frac{\int_{\Omega^*} |u_1^*(x)|^2 dx}{\int_{\Omega^*} \int_{\Omega^*} u_1^*(y)\theta(x-y) u_1^*(x) dy dx} \geq$$

$$\inf_{\vartheta \in L^2(\Omega^*)} \frac{\int_{\Omega^*} |\vartheta(x)|^2 dx}{\int_{\Omega^*} \int_{\Omega^*} \vartheta(y)\theta(x - y)\vartheta(x) dy dx} = \mu_1(\Omega^*).$$

Theorem 1 is proved.

3Krahn-Szego theorem. In this section we are interested in minimizing the second characteristic number of the Newton potential N among open sets of given volume. As in case of the Dirichlet Laplacian, the minimizer is no longer one ball, but two!

Theorem 2.*The minimum of $\mu_2(\Omega)$ among bounded open sets of R^d with given volume is achieved by the union of two identical balls.*

Similar result for the Dirichlet Laplacian is called the Krahn-Szego theorem. See, for example, [1] for further references.

Proof of Theorem 2. Lemma 1 says that among eigenfunctions of N only the first eigenfunction is positive

$$u_1(x) > 0, \forall x \in \Omega.$$

Therefore,

$$u_2(x) > 0, \forall x \in \Omega^+ \subset \Omega, \Omega^+ \neq \{0\}.$$
$$u_2(x) < 0, \forall x \in \Omega^- \subset \Omega, \Omega^- \neq \{0\}.$$

We have

$$u_2(x) = \mu_2(\Omega) \int_\Omega \varepsilon_d(x-y) u_2(y) dy, x \in \Omega.$$

Taking

$$u_2^+(x) = \begin{cases} u_2(x) \; in \; \Omega^+, \\ 0 \; otherwise, \end{cases} \tag{12}$$

and

$$u_2^-(x) = \begin{cases} u_2(x) \; in \; \Omega^-, \\ 0 \; otherwise, \end{cases}$$

we obtain

$$u_2(x) = \mu_2(\Omega) \int_{\Omega^+} \varepsilon_d(x-y) u_2^+(y) dy + \mu_2(\Omega) \int_{\Omega^-} \varepsilon_d(x-y) u_2^-(y) dy, x \in \Omega.$$

Multiplying by $u_2^+(x)$ and integrating over Ω^+ we get

$$\int_{\Omega^+} |u_2^+(x)|^2 dx = \mu_2(\Omega) \int_{\Omega^+} u_2^+(x) \int_{\Omega^+} \varepsilon_d(x-y) u_2^+(y) dy dx +$$
$$\mu_2(\Omega) \int_{\Omega^+} u_2^+(x) \int_{\Omega^-} \varepsilon_d(x-y) u_2^-(y) dy dx, \qquad x \in \Omega.$$

The second term in the right hand sight is negative as we know sign of all integrants. Therefore, one has

$$\int_{\Omega^+} |u_2^+(x)|^2 dx \leq \mu_2(\Omega) \int_{\Omega^+} u_2^+(x) \int_{\Omega^+} \varepsilon_d(x-y) u_2^+(y) dy dx,$$

that is,

$$\frac{\int_{\Omega^+} |u_2^+(x)|^2 dx}{\int_{\Omega^+} u_2^+(x) \int_{\Omega^+} \varepsilon_d(x-y) u_2^+(y) dy dx} \leq \mu_2(\Omega)$$

From here by using the variational principle one obtains

$$\mu_2(\Omega^+) = inf_{\vartheta \in L^2(\Omega^+)} \frac{\int_{\Omega^+} |\vartheta(x)|^2 dx}{\int_{\Omega^+} \vartheta(x) \int_{\Omega^+} \varepsilon_d(x-y) \vartheta(y) dy dx}$$
$$\leq \frac{\int_{\Omega^+} |u_2^+(x)|^2 dx}{\int_{\Omega^+} u_2^+(x) \int_{\Omega^+} \varepsilon_d(x-y) u_2^+(y) dy dx} \leq \mu_2(\Omega).$$

Similarly, we get

$$\mu_1(\Omega^-) \leq \mu_2(\Omega).$$

So we have

$$\mu_1(\Omega^+) \leq \mu_2(\Omega), \mu_1(\Omega^-) \leq \mu_2(\Omega). \tag{13}$$

We now introduce $B^+$and B^-, balls of the same volume as Ω^+ and Ω^-, correspondingly. According to Theorem 1, we have

$$\mu_1(B^+) \le \mu_1(\Omega^+), \qquad \mu_1(B^-) \le \mu_1(\Omega^-). \tag{14}$$

Let us introduce a new open set $\tilde{\Omega}$ defined as $\tilde{\Omega} = B^+ \cup B^-$. Since $\tilde{\Omega}$ is disconnected, we obtain its eigenvalues by gathering and reordering the eigenvalues of $B^+$and B^-. Therefore,

$$\mu_2(\tilde{\Omega}) \le \max(\mu_1(B^+), \mu_1(B^-)).$$

According to (13) and (14) we have

$$\mu_2(\tilde{\Omega}) \le \max(\mu_1(B^+), \mu_1(B^-)) \le \max(\mu_1(\Omega^+), \mu_1(\Omega^-)) \le \mu_2(\Omega).$$

This shows that, in any case, the minimum of μ_2 is to be sought among the union of balls. But, if the two balls would have different radii, we would decrease the second eigenvalue by shrinking the largest one and dilating the smaller one (without changing the total volume). Therefore, the minimum is achieved by the union of two identical balls.

Theorem 2 isproved.

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НЬЮТОН ПОТЕНЦИАЛЫНЫҢ ЕКІНШІ СИПАТТЫҚ САНЫ ЖАЙНДА

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Аннотация. Жұмыста Ньютон потенциалының екінші сипаттық саны берілген көлемді R^d-дегі шенелген ашық жиындар арасында өзінін ең аз шамасын бір-келкі екі шардың біргуінде қабылдайтынын дәлелдейміз. Ньютон потенциалы Лапласианның локальды емес шекаралық есебіне қатасты болып табылатындықтан алынатын нәтижелер Лапласианның локальды емес шекаралық есебіне екінші меншікті мәнінеде тиісті.

НЬЮТОН ПОТЕНЦИАЛЫНЫҢ ЕКІНШІ СИПАТТЫҚ САНЫ ЖАЙНДА

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