

N E W S

OF THE NATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN

PHYSICO-MATHEMATICAL SERIES

ISSN 1991-346X

Volume 3, Number 301 (2015), 91 – 96

CLARKSON WEAK MAJORIZATION INEQUALITIES FOR τ -MEASURABLE OPERATORS

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Key words: Clarkson inequality, τ -measurable operator, von Neumann algebra, submajorization.

Abstract. Let (M, τ) be a semi-finite von Neumann algebra and f be a nonnegative function on $[0, \infty)$ with $f(0) = 0$. Let x_1, x_2, \dots, x_n be τ -measurable operators and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be positive real numbers such that $\sum_{j=1}^n \alpha_j = 1$. We have the following results.

(1) If $g(t) = f(\sqrt{t})$ is convex, then

$$f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right) \leq \sum_{j=1}^n \alpha_j f(|x_j|).$$

(2) If $h(t) = f(\sqrt{t})$ is concave, then

$$\sum_{j=1}^n \alpha_j f(|x_j|) \leq f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right).$$

1. Introduction. Let M_n be von Neumann algebra of $n \times n$ complex matrices, and let M_n^+ be positive part of M_n . Hirzallah and Kittaneh in [5] proved the following noncommutative Clarkson inequalities for n -tuples of operators: Let $\|\cdot\|$ be a unitarily norm, $A_0, \dots, A_{n-1} \in M_n^+$ and $\alpha_0, \dots, \alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1} \alpha_j = 1$.

(1) If f be a nonnegative function on $[0, \infty)$ such that $f(0) = 0$ and $g(t) = f(\sqrt{t})$ is convex on $[0, \infty)$, then

$$\left\| f\left(\sum_{j=0}^{n-1} \alpha_j A_j\right) + \sum_{0 \leq j < k \leq n-1} f(\sqrt{\alpha_j \alpha_k} |A_j - A_k|) \right\| \leq \left\| \sum_{j=0}^{n-1} \alpha_j f(|A_j|) \right\|.$$

(2) If f be a nonnegative function on $[0, \infty)$ such that $g(t) = f(\sqrt{t})$ is concave on $[0, \infty)$, then

$$\left\| \sum_{j=0}^{n-1} \alpha_j f(|A_j|) \right\| \leq \left\| f\left(\sum_{j=0}^{n-1} \alpha_j A_j\right) + \sum_{0 \leq j < k \leq n-1} f(\sqrt{\alpha_j \alpha_k} |A_j - A_k|) \right\|.$$

The purpose of this paper is to extend the above results to n -tuples of τ -measurable operators.

This paper is organized as follows. Section 2 contains some preliminary definitions.

In section 3, we proved the weak majorization type of Clarkson inequalities for n tuples of τ -measurable operators.

2. Preliminaries. Throughout this paper, we denote by M a semi-finite von Neumann algebra on the Hilbert space H with a normalized normal faithful finite trace τ . The closed densely defined linear operator x in H with domain $D(x)$ is said to be affiliated with M if and only if $u^*xu = x$ for all unitary u which belong to the commutant M' of M . If x is affiliated with M , the x said to be τ -measurable if for every $\varepsilon > 0$ there exists a projection $e \in M$ such that $e(H) \subseteq D(x)$ and $\tau(e^\perp) < \varepsilon$ (where for any projection e we let $e^\perp = 1 - e$). The set of all τ -measure operators will be denoted by $L_0(M)$. The set $L_0(M)$ is a $*$ -algebra with sum and product being the respective closure of the algebraic sum and product. Let $P(M)$ be the lattice of projections of M . The sets

$$N(\varepsilon, \delta) = \left\{ x \in L_0(M) : \exists e \in P(M) \text{ such that } \|xe\| < \varepsilon, \tau(e^\perp) < \delta \right\}$$

$(\varepsilon, \delta > 0)$ from a base at 0 for an metrizable Hausdorff topology in $L_0(M)$ called the measure topology. Equipped with the measure topology, $L_0(M)$ is a complete topological $*$ -algebra (see [6]). For $x \in L_0(M)$, the generalized singular value function $\mu(x)$ of x is defined by

$$\mu_s(x) = \inf \left\{ \|xe\| : e \in P(M), \tau(e^\perp) \leq s \right\} \quad (s \geq 0).$$

If $x, y \in L_0(M)$, then we say that x is submajorized by y and write $x \preceq y$ if and only if

$$\int_0^t \mu_s(x) ds \leq \int_0^t \mu_s(y) ds, \quad \forall t > 0.$$

As Proposition 4.6 in [3], we obtain the following result.

Lemma 2.1. Let f be a continuous increasing function on \mathbf{R}_+ with $f(0) = 0$. Let x_1, x_2, \dots, x_n be positive τ -measurable operators and let a_1, a_2, \dots, a_n be positive elements in M with $\sum_{j=1}^n a_j^* a_j \leq 1$.

(1) When f is convex, we have

$$\mu_s \left[f \left(\sum_{j=1}^n a_j^* x_j a_j \right) \right] \leq \mu_s \left[\sum_{j=1}^n a_j^* f(x_j) a_j \right], \quad s > 0. \quad (2.1)$$

(2) When f is concave, we have

$$\mu_s \left[\sum_{j=1}^n a_j^* f(x_j) a_j \right] \leq \mu_s \left[f \left(\sum_{j=1}^n a_j^* x_j a_j \right) \right], \quad s > 0. \quad (2.2)$$

The following lemma is a well-known result (see [2], Theorem 5.3.).

Lemma 2.2. Let x_1, x_2, \dots, x_n be positive τ -measurable operators.

(1) If $g: [0, \infty) \rightarrow [0, \infty)$ be a convex function with $g(0) = 0$. Then

$$\sum_{j=1}^n g(x_j) \leq g \left(\sum_{j=1}^n x_j \right). \quad (2.3)$$

(2) If $h: [0, \infty) \rightarrow [0, \infty)$ be a concave function. Then

$$h \left(\sum_{j=1}^n x_j \right) \leq \sum_{j=1}^n h(x_j). \quad (2.4)$$

3. Main result. Let x_1, x_2, \dots, x_n be positive τ -measurable operators and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be positive real numbers such that $\sum_{j=1}^n \alpha_j = 1$. Then

$$\left| \sum_{j=1}^n \alpha_j x_j \right|^2 + \sum_{1 \leq j < k \leq n} \alpha_j \alpha_k |x_j - x_k|^2 = \sum_{j=1}^n \alpha_j |x_j|^2. \quad (3.1)$$

Theorem 3.1. Let x_1, x_2, \dots, x_n be τ -measurable operators and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be positive real numbers such that $\sum_{j=1}^n \alpha_j = 1$ and $f: [0, \infty) \rightarrow [0, \infty)$ such that $f(0) = 0$ and $g(t) = f(\sqrt{t})$ is convex on $[0, \infty)$. Then

$$f \left(\left| \sum_{j=1}^n \alpha_j x_j \right| \right) + \sum_{1 \leq j < k \leq n} f \left(\sqrt{\alpha_j \alpha_k} |x_j - x_k| \right) \leq \sum_{j=1}^n \alpha_j f(|x_j|). \quad (3.2)$$

Proof. By (2.1), (2.3) and (3.1), we obtain that

$$\begin{aligned} \sum_{j=1}^n \alpha_j f(|x_j|) &= \sum_{j=1}^n \alpha_j g(|x_j|^2) \geq g \left(\sum_{j=1}^n \alpha_j |x_j|^2 \right) \\ &= g \left(\left| \sum_{j=1}^n \alpha_j x_j \right|^2 + \sum_{1 \leq j < k \leq n} \alpha_j \alpha_k |x_j - x_k|^2 \right) \end{aligned}$$

$$\begin{aligned}
 &\succeq g\left(\left|\sum_{j=1}^n \alpha_j x_j\right|^2\right) + \sum_{1 \leq j < k \leq n} g(\alpha_j \alpha_k |x_j - x_k|^2) \\
 &= f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f(\sqrt{\alpha_j \alpha_k} |x_j - x_k|).
 \end{aligned}$$

□

Theorem 3.2. Let x_1, x_2, \dots, x_n be τ -measurable operators and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be positive real numbers such that $\sum_{j=1}^n \alpha_j = 1$ and $f: [0, \infty) \rightarrow [0, \infty)$ such that $h(t) = f(\sqrt{t})$ is concave on $[0, \infty)$.

Then

$$\sum_{j=1}^n \alpha_j f(|x_j|) \preceq f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f(\sqrt{\alpha_j \alpha_k} |x_j - x_k|). \quad (3.3)$$

Proof. We may assume $f(0) = 0$. Using (2.2), (2.4) and (3.1) we get that

$$\begin{aligned}
 \sum_{j=1}^n \alpha_j f(|x_j|) &= \sum_{j=1}^n \alpha_j h(|x_j|^2) \preceq h\left(\sum_{j=1}^n \alpha_j |x_j|^2\right) \\
 &= h\left(\left|\sum_{j=1}^n \alpha_j x_j\right|^2 + \sum_{1 \leq j < k \leq n} \alpha_j \alpha_k |x_j - x_k|^2\right) \\
 &\preceq h\left(\left|\sum_{j=1}^n \alpha_j x_j\right|^2\right) + \sum_{1 \leq j < k \leq n} h(\alpha_j \alpha_k |x_j - x_k|^2) \\
 &= f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f(\sqrt{\alpha_j \alpha_k} |x_j - x_k|) \quad . \quad \square
 \end{aligned}$$

Specializing Theorems 3.1 and 3.2 to the functions $f(t) = t^p$ ($2 \leq p < \infty$) and $f(t) = t^p$ ($0 \leq p \leq 2$).

Corollary 3.1. Let x_1, x_2, \dots, x_n be τ -measurable operators and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be positive real numbers such that $\sum_{j=1}^n \alpha_j = 1$. Then

$$\left|\sum_{j=1}^n \alpha_j x_j\right|^p + \sum_{1 \leq j < k \leq n} (\alpha_j \alpha_k)^{\frac{p}{2}} |x_j - x_k|^p \preceq \sum_{j=1}^n \alpha_j |x_j|^p$$

for $2 \leq p < \infty$, and

$$\sum_{j=1}^n \alpha_j |x_j|^p \preceq \left|\sum_{j=1}^n \alpha_j x_j\right|^p + \sum_{1 \leq j < k \leq n} (\alpha_j \alpha_k)^{\frac{p}{2}} |x_j - x_k|^p$$

for $0 \leq p \leq 2$.

Applying Corollary 3.1 for the trace norm $\|\cdot\|_1$, we have the following result.

Corollary 3.2. Let x_1, x_2, \dots, x_n be τ -measurable operators and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be positive real numbers such that $\sum_{j=1}^n \alpha_j = 1$. Then

$$\left\| \sum_{j=1}^n \alpha_j x_j \right\|_p^p + \sum_{1 \leq j < k \leq n} (\alpha_j \alpha_k)^{\frac{p}{2}} \|x_j - x_k\|_p^p \leq \sum_{j=1}^n \alpha_j \|x_j\|_p^p$$

for $2 \leq p < \infty$, and

$$\sum_{j=1}^n \alpha_j \|x_j\|_p^p \leq \left\| \sum_{j=1}^n \alpha_j x_j \right\|_p^p + \sum_{1 \leq j < k \leq n} (\alpha_j \alpha_k)^{\frac{p}{2}} \|x_j - x_k\|_p^p$$

for $0 \leq p \leq 2$. In particular

$$\left\| \sum_{j=1}^n x_j \right\|_p^p + \sum_{1 \leq j < k \leq n} \|x_j - x_k\|_p^p \leq n^{p-1} \sum_{j=1}^n \|x_j\|_p^p$$

for $2 \leq p < \infty$, and

$$n^{p-1} \sum_{j=1}^n \|x_j\|_p^p \leq \left\| \sum_{j=1}^n x_j \right\|_p^p + \sum_{1 \leq j < k \leq n} \|x_j - x_k\|_p^p$$

for $0 \leq p \leq 2$.

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τ-ӨЛШЕМДІ ОПЕРАТОРЛАР ҮШИН КЛАРКСОННЫҢ ӘЛСІЗ МАЖОРЛАНГАН ТЕНСІЗДІКТЕРИ

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Тірек сөздер: Кларксон тенсіздіктері, τ -өлшемді оператор, Фон Нейман алгебрасы, субмажорланған.

Аннотация. (M, τ) жартылай ақырлы фон Нейман алгебрасы және $f \in C[0, \infty)$ тегі теріс емес функция болсын. x_1, x_2, \dots, x_n дер τ -өлшемді операторлар және $\alpha_1, \alpha_2, \dots, \alpha_n$ он накты сандары $\sum_{j=1}^n \alpha_j = 1$ болсын. Біз келесі нәтижелерді аламыз.

(1) Егер $g(t) = f(\sqrt{t})$ дәнес болса, онда

$$f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right) \leq \sum_{j=1}^n \alpha_j f(|x_j|).$$

(2) Егер $h(t) = f(\sqrt{t})$ ойыс болса, онда

$$\sum_{j=1}^n \alpha_j f(|x_j|) \leq f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right).$$

НЕРАВЕНСТВА СЛАБО МАЖОРИЗАЦИОННЫЕ КЛАРКСОНА ДЛЯ τ -ИЗМЕРИМЫХ ОПЕРАТОРОВ

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Ключевые слова: неравенства Кларксона, τ -измеримый оператор, алгебра Фон Неймана, субмажоризация.

Аннотация. Пусть (M, τ) полу конечная алгебра Фон Неймана и f неотрицательная функция на $[0, \infty)$ с $f(0) = 0$. Пусть x_1, x_2, \dots, x_n τ -измеримых операторов и $\alpha_1, \alpha_2, \dots, \alpha_n$ положительные вещественные числа такие, что $\sum_{j=1}^n \alpha_j = 1$. Мы получили следущие результаты.

(1) Если $g(t) = f(\sqrt{t})$ является выпуклым, то

$$f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right) \leq \sum_{j=1}^n \alpha_j f(|x_j|).$$

(2) Если $h(t) = f(\sqrt{t})$ является вогнутой, то

$$\sum_{j=1}^n \alpha_j f(|x_j|) \leq f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right).$$

Поступила 25.02.2015 г.