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**SUBADDITIVITY WEAK MAJORIZATION INEQUALITIES
FOR τ -MEASURABLE OPERATORS**

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Abstract. Let (M, τ) be a semi-finite von Neumann algebra and let and $f : [0, \infty) \rightarrow [0, \infty)$ is increasing concave function such that $f(0) = 0$. We have the following results:

- If x, y are τ -measurable operators, then $f(|x+y|) \leq f(|x|) + f(|y|)$.
- If x is self-adjoint τ -measurable operator and $z \in M$ is expansive operator, then $f(|z^*xz|) \leq z^*f(|x|)z$.

1. Introduction. Let M_n be von Neumann algebra of $n \times n$ complex matrices, and let M_n^+ be positive part of M_n . Bourin in [3] proved the following a matrix subadditivity inequality for symmetric norms: Let $f : [0, \infty) \rightarrow [0, \infty)$ be concave.

- If A and B be normal matrices. Then, for all symmetric norms

$$\|f(A+B)\| \leq \|f(A)\| + \|f(B)\|.$$

- If A be normal and let Z be expansive. Then, for all symmetric norms

$$\|f(Z^*AZ)\| \leq \|Z^*f(A)Z\|.$$

The purpose of this paper is to extend the above results to n -tuples of τ -measurable operators.

This paper is organized as follows. Section 2 contains some preliminary definitions.

In section 3, we proved the weak majorization type of subadditivity inequalities for n tuples of τ -measurable operators.

2. Preliminaries. Throughout this paper, we denote by M a semi-finite von Neumann algebra on the Hilbert space H with a normalized normal faithful finite trace τ . The closed densely defined linear operator x in H with domain $D(x)$ is said to be affiliated with M if and only if $u^*xu = x$ for all unitary u which belong to the commutant M' of M . If x is affiliated with M , the x said to be τ -measurable if for every $\varepsilon > 0$ there exists a projection $e \in M$ such that $e(H) \subseteq D(x)$ and $\tau(e^\perp) < \varepsilon$ (where for any projection e we let $e^\perp = 1 - e$). The set of all τ -measure operators will be denoted by $L_0(M)$. The set $L_0(M)$ is a $*$ -algebra with sum and product being the respective closure of the algebraic sum and product. Let $P(M)$ be the lattice of projections of M . The sets

$$N(\varepsilon, \delta) = \{x \in L_0(M) : \exists e \in P(M) \text{ such that } \|xe\| < \varepsilon, \tau(e^\perp) < \delta\}$$

$(\varepsilon, \delta > 0)$ from a base at 0 for an metrizable Hausdorff topology in $L_0(M)$ called the measure topology. Equipped with the measure topology, $L_0(M)$ is a complete topological $*$ -algebra (see [6]). For $x \in L_0(M)$, the generalized singular value function $\mu(x)$ of x is defined by

$$\mu_s(x) = \inf\{\|xe\| ; e \in P(M), \tau(e^\perp) \leq s\}, s \geq 0.$$

If $x, y \in L_0(M)$, then we say that x is submajorized by y and write $x \leq y$ if and only if

$$\int_0^t \mu_s(x) ds \leq \int_0^t \mu_s(y) ds, \quad t \in [0, 1].$$

The following lemma is a well-known result (see [4], Theorem 5.2.).

Lemma 2.1. Let x, y be positive τ -measurable operators and $f : [0, \infty) \rightarrow [0, \infty)$ be a concave function. Then

$$f(x+y) \leq f(x) + f(y). \tag{1}$$

In particular, Theorem 3.2.8., In [2], we obtain the following result.

Lemma 2.2. Let $f : [0, \infty) \rightarrow [0, \infty)$ be a concave function, and let $z \in M$ be an expansive operator. Then

$$f(z^*xz) \leq z^*f(x)z, \quad \forall x \in L_0(M)^+. \tag{2}$$

3. Main result

Theorem 3.12 Let x, y be τ -measurable operators and let and $f : [0, \infty) \rightarrow [0, \infty)$ is increasing concave function such that $f(0) = 0$. Then

$$f(|x+y|) \leq f(|x|) + f(|y|). \quad (3)$$

Proof. Lemma 4.3 in [5] there exist partial isometries u, v in M such that

$$|x+y| \leq u|x|u^* + v|y|v^*$$

Applying Lemma 2.5. (iii) in [5].

$$\mu_s(|x+y|) \leq \mu_s(u|x|u^* + v|y|v^*)$$

Since continuous increasing of f , we have that

$$f(\mu_s(|x+y|)) \leq f(\mu_s(u|x|u^* + v|y|v^*)).$$

By Lemma 2.5. (iv) in [5], we get

$$\mu_s(f(|x+y|)) \leq \mu_s(f(u|x|u^* + v|y|v^*)).$$

Using inequality (1), we obtain that

$$f(|x+y|) \leq f(u|x|u^* + v|y|v^*) \leq f(u|x|u^*) + f(v|y|v^*).$$

By Lemma 2.5. (iii) in [5], we take

$$\mu_s(u|x|u^*) \leq \|u\| \|u^*\| \mu_s(|x|) \leq \mu_s(|x|).$$

Since continuous increasing of f

$$f(\mu_s(u|x|u^*)) \leq f(\mu_s(|x|)).$$

By Lemma 2.5. (iv) in [5], we have that

$$\mu_s(f(u|x|u^*)) \leq \mu_s(f(|x|))$$

so

$$f(u|x|u^*) \leq f(|x|)$$

and we obtain the same inequality as a following:

$$f(v|y|v^*) \leq f(|y|).$$

Hence

$$f(|x+y|) \leq f(|x|) + f(|y|).$$

Corollary 3.1. 3 Let x, y be normal τ -measurable operators and let and $f : [0, \infty) \rightarrow [0, \infty)$ is increasing concave function such that $f(0) = 0$. Then

$$f(|x+y|) \leq f(|x|) + f(|y|).$$

Theorem 3.2. 4 Let x be self-adjoint τ -measurable operators and let and $f : [0, \infty) \rightarrow [0, \infty)$ is increasing concave function such that $f(0) = 0$. Then

$$f(|z^*xz|) \leq z^*f(|x|)z. \quad (4)$$

Proof. It is clear that

$$x \leq |x|.$$

Then by using Proposition 4.5 (iii) [7]

$$z^*xz \leq z^*|x|z.$$

By Lemma 2.5. (ii),(iii) in [5], we take

$$\mu_s(|z^*xz|) = \mu_s(z^*xz) \leq \mu_s(z^*|x|z).$$

Since continuous increasing of f , we get

$$f(\mu_s(|z^*xz|)) \leq f(\mu_s(z^*|x|z)).$$

Applying Lemma 2.5. (iv) in [5], we have that

$$\mu_s(f(|z^*xz|)) \leq \mu_s(f(z^*|x|z)).$$

By continuously of integral and inequality (2), we obtain

$$f(|z^*xz|) \leq f(z|x|z) \leq z^*f(|x|)z.$$

Theorem 5 3.3. Let x be τ -measurable operators and let and $f : [0, \infty) \rightarrow [0, \infty)$ is increasing concave function such that $f(0) = 0$. Then

$$\begin{pmatrix} f(|z^*xz|) & 0 \\ 0 & f(|z^*x^*z|) \end{pmatrix} \preceq \begin{pmatrix} z^*f(|x|)z & 0 \\ 0 & z^*f(|x^*|)z \end{pmatrix}. \quad (5)$$

Proof. Applying Theorem 3.2 to the Hermitian operators

$$\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}$$

we obtain

$$f\left(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}^*\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\right) \preceq \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}^* f\left(\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}\right) \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

Hence

$$\begin{aligned} & \begin{pmatrix} f(|z^*xz|) & 0 \\ 0 & f(|z^*x^*z|) \end{pmatrix} \\ &= f\left(\begin{pmatrix} |z^*xz| & 0 \\ 0 & |z^*x^*z| \end{pmatrix}\right) = f\left(\begin{pmatrix} 0 & z^*x^*z \\ z^*xz & 0 \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}^*\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\right) \\ &\preceq \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}^* f\left(\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}\right) \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \\ &= \begin{pmatrix} z^* & 0 \\ 0 & z^* \end{pmatrix} f\left(\begin{pmatrix} |x| & 0 \\ 0 & |x^*| \end{pmatrix}\right) \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \\ &= \begin{pmatrix} z^* & 0 \\ 0 & z^* \end{pmatrix} \begin{pmatrix} f(|x|) & 0 \\ 0 & f(|x^*|) \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \\ &= \begin{pmatrix} z^*f(|x|)z & 0 \\ 0 & z^*f(|x^*|)z \end{pmatrix} \end{aligned}$$

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**τ-ӨЛШЕМДІ ОПЕРАТОРЛАР ҮШИН
СУБАДДИТИВТІ ӘЛСІЗ МАЖОРЛАНГАН ТЕҢСІЗДІКТЕР**

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ҚР ҰҒА Математика және математикалық моделдеу институты,
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Тірек сөздер: субаддитивті теңсіздіктері, τ -өлшемді оператор, кенейтуші оператор, Фон Нейман алгебрасы, субмажорланған.

Аннотация. (M, τ) жартылай ақырлы фон Нейман алгебрасы және $f : [0, \infty) \rightarrow [0, \infty)$ $f(0) = 0$ болатын еспелі ойыс функция болсын. Біз келесі нәтижелерді аламыз.

(1) Егер x, y тер τ -өлшемді операторлар болса, онда

$$f(|x+y|) \leq f(|x|) + f(|y|).$$

(2) Егер x өзіне-өзі түйіндес τ -өлшемді оператор және $z \in M$ кенейтуші оператор болса, онда

$$f(|z^* x z|) \leq z^* f(|x|) z.$$

**СУБАДДИТИВНОСТЬ СЛАБО МАЖОРИЗАЦИОННЫХ НЕРАВЕНСТВ
ДЛЯ τ -ИЗМЕРИМЫХ ОПЕРАТОРОВ**

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Ключевые слова: субаддитивное неравенство, τ -измеримый оператор, расширяющий оператор, алгебра Фон Неймана, субмажоризация.

Аннотация. Пусть (M, τ) полу конечная алгебра Фон Неймана и $f : [0, \infty) \rightarrow [0, \infty)$ возрастающая вогнутая функция с $f(0) = 0$. Мы получили следующие результаты.

(1) Если x, y τ -измеримые операторы, то

$$f(|x+y|) \leq f(|x|) + f(|y|).$$

(2) Если x самосопряженный τ -измеримый оператор и $z \in M$ расширяющий оператор, то

$$f(|z^* x z|) \leq z^* f(|x|) z.$$

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