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ON AN INEQUALITY FOR SCHATTEN p-NORMS

D.Suragan

Institute of Mathematics and Mathematical Modeling
Almaty, Kazakhstan

suragan@list.ru

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Abstract. In this note, we prove an abstract inequality for Schatten p-norm of compact operators. Our result gives an answer to B. Simon's conjecture on Schatten p-norm domination of integral operators in terms of completely monotonic functions.

Introduction

Let H be a separable Hilbert space. We denote the class of compact operators $P: H \to H$ by $S^\infty(H)$. Recall that the singular values $\{s_n\}$ of $P \in S^\infty(H)$ are the eigenvalues of the positive operator $(P^*P)^{1/2}$ (see e.g. Gohberg and Krein [7]). The Schatten p-classes are defined as

$$S^p(H) := \{P \in S^\infty(H) : \{s_n\} \in \ell^p\}, \quad 1 \le p < \infty.$$

In $S^p(H)$ the Schatten p-norm of the operator P is defined as

$$\|P\|_p := \left(\sum_{n=1}^{\infty} s_n^p\right)^{\frac{1}{p}}, \quad 1 \le p < \infty. \tag{1.1}$$

For $p = \infty$, we can set

$$\|P\|_\infty := \|P\|$$

to be the operator norm of P on H. In B. Simon's Trace ideals and their applications book (p. 24, [17]), there are formulated two conjectures related to Schatten p-norm dominations. In Theorem 2.13 ([17], the first edition of the book was published in 1979) it is shown that the abstract notion of domination implies the Schatten p-norms of even integer order satisfy a domination inequality. And the first conjecture was that Theorem 2.13 is valid for not only even integer order, but also it is valid for all Schatten p-norms. However, Peller [18] has shown, using Hankel operators, that this conjecture is wrong; see Addendum E in the book [17]. Our main result (2.1) gives an answer to B. Simon's second conjecture on Schatten p-norm domination of integral operators in terms of completely monotonic functions.

B. Simon's conjecture: *If f^* is the symmetric decreasing rearrangement of f, then for $p \ge 2$, the Schatten p-norm of the operator with kernel $f^*(x-y)g^*(y)$ dominates the Schatten p-norm of the one with kernel $f(x-y)g(y)$.*

The positivity of a certain inverse Laplace transform and properties of competely monotonic functions play a key role in our proof. A function F is said to be completely monotonic in $(0,\infty)$, if F has derivatives of all orders and satisfies

$$(-1)^k F^k(\varsigma) \geq 0, \quad k = 0,1,2,..., \tag{1.2}$$

for all $\varsigma > 0$. If the function F can be extended to be continuous at $\varsigma = 0$ it is said to be completely monotonic in $[0,\infty)$. The definition of a completely monotonic function was introduced by F. Hausdorff [8]. In his work these functions was called as 'total monotonic' functions. Later Bernstein proved (Bernstein's theorem) that F is completely monotonic if and only it has the representation

$$F(\varsigma) = \int_0^\infty e^{-\varsigma t} d\mu(t),$$

where $\mu(t)$ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $\varsigma > 0$. Dubourdieu [5] proved that if a completely monotonic function F is not a constant then strict inequality holds in (1.2) . For more discussions on completely monotonic functions we refer [6] and references therein.

In Section (2) we present the main result of this paper. Its proof will be given in Section (3).

Main result and examples.

As outlined in the introduction, let H be a separable Hilbert space. Let A,B operators belong to the Schatten class $S^q(H)$. Define the function

$$F_{AB}(\varsigma) := \sum_{j=1}^{\infty} \frac{1}{\mu_j^{q-1}(A)(\mu_j(A)+\varsigma)} - \frac{1}{\mu_j^{q-1}(B)(\mu_j(B)+\varsigma)}, \ \varsigma \geq 0,$$

where $\mu_j(A)$ and $\mu_j(B)$ are the inverses of the jth singular values of operators A and B, respectively. Clearly, the function F_{AB} is analytic in $[0,\infty)$.

Theorem 2.1. *Let* $A,B \in S^q(H)$. *If the function* F_{AB} *is completely monotonic in* $[0,\infty)$, *then*

$$\|A\|_p \geq \|B\|_p \tag{2.1}$$

for all $p \geq q$.

Let us take

$$Au := \int_{R^d} f(x-y)g(y)u(y)dy,$$

$$Bu := \int_{R^d} f^*(x-y)g^*(y)u(y)dy.$$

Then from Theorem 2.1 we see that B. Simon's conjecture is correct for any Schatten p-class if the corresponding F_{AB} function is completely monotonic. Obviously, an essential question is: How can we check that the function F_{AB} is completely monotonic? An answer depends on the formula of F_{AB}. One may use Bernstein's theorem as we mentioned in the introduction. However, this theorem is not always applicable. Therefore, one might choose other methods (cf. [11]). Let f be strictly symmetric decreasing function, that is, the function $f(|x|)$ is a real, positive and decreasing, i.e. that the function $f:[0,\infty]\to R$ satisfies

$$f(\rho)>0 \quad for\ any \quad \rho\geq 0, \tag{2.2}$$

and

$$f(\rho_1)>f(\rho_2) \quad if \quad \rho_1<\rho_2, \tag{2.3}$$

let g be a characteristic function of Euclidean domain $\Omega\subset R^d$ i.e.

$$g(x)=\begin{cases}1, x\in\Omega,\\ 0, otherwise\end{cases} \tag{2.4}$$

This implies that f and g do not change their formulae under the symmetric-decreasing rearrangement, see e.g. Lieb and Loss [10]. Therefore, we have

$$Au:=\int_{\Omega} f(|x-y|)u(y)dy,$$

$$Bu:=\int_{\Omega^*} f(|x-y|)u(y)dy,$$

where Ω^* is the symmetric rearrangement of Ω, that is, Ω^* is a centred ball with $|\Omega^*|=|\Omega|$. Here by $|\Omega|$ we denote the Lebesque measure of Ω.

If one shows that F_{AB} is completely monotonic, then in this special case B. Simon's conjecture is correct.

Here we give some related examples for further motivations. In a bounded open domain $\Omega\subset R^3$ let us consider the Newton potential on $L^2(\Omega)$ as an example, i.e. let

$$N_{\Omega}f(x):=\int_{\Omega}\frac{1}{4\pi|x-y|}f(y)dy, \quad f\in L^2(\Omega), \tag{2.5}$$

where $|x-y|$ is the standard Euclidian distance between x and y. Note that the operator N_{Ω} is the inverse to the Laplacian (see [9]). It is not difficult to show that the Newton potential operator is a Hilbert-Schmidt operator. It can be also followed from the properties of the N_{Ω} operator kernel see e.g. the criteria for Schatten classes in terms of the regularity of the kernel in [4]. We obtain:

Example 2.2 [13] *Let $B\equiv U$ be the unit 3-ball. Then by Theorem (2.1) we have*

$$\|N_{\Omega}\|_p\leq\|N_U\|_p=\left(\sum_{l=0}^{\infty}\sum_{m=1}^{\infty}\frac{2l+1}{j^{2p}_{l-\frac{1}{2},m}}\right)^{\frac{1}{p}}, \tag{2.6}$$

for any integer $2\leq p<\infty$ and any bounded open domain Ω with $|\Omega|=|U|$. Here j_{km} denotes the m^{th} positive zero of the Bessel function J_k of the first kind of order k.

Example 2.3 [13] *Let* $B \equiv U$ *be the unit 3-ball. In particular, for the Hilbert-Schmidt norm we have*

$$\|N_\Omega\|_2 \le \|N_U\|_2 = \sqrt{\frac{7}{48}}, \tag{2.7}$$

for any bounded open domain Ω with $|\Omega| = |U|$.

The above examples justify the constants for the Newton potential that were also announced in [15]. We omit the routine technical calculation.

Lemma 3.1. *If the function* F_{AB} *is completely monotonic in* $[0,\infty)$, *then*

$$\sum_{j=1}^{\infty} \frac{e^{-\mu_j(A)t}}{\mu_j^{p-1}(A)} \ge \sum_{j=1}^{\infty} \frac{e^{-\mu_j(B)t}}{\mu_j^{p-1}(B)}, \quad \forall t > 0, \tag{3.1}$$

for any $p \ge q$

Proof of Lemma (3.1) .Since F_{AB} is completely monotonic it is non-negative, that is,

$$\sum_{j=1}^{\infty} \frac{1}{\mu_j^{q-1}(A)(\mu_j(A)+\varsigma)} \ge \sum_{j=1}^{\infty} \frac{1}{\mu_j^{q-1}(B)(\mu_j(B)+\varsigma)} \tag{3.2}$$

for all $\varsigma \ge 0$.

Let L be the Laplace transform

$$L\{f(t)\}(\varsigma) = \int_0^\infty e^{-\varsigma t} f(t)dt.$$

Using the inverse Laplace transform (exponential decay) we have

$$L^{-1}\{\frac{1}{\mu_j + \varsigma}\} = e^{-\mu_j t}, t > 0. \tag{3.3}$$

for $\varsigma > -\mu_j$. By applying L^{-1} to both sides of (3.2) we obtain (3.1) (see Lemma (3.3)).

One might have a question concerning the proof of Lemma (3.1), that is, why does the inverse Laplace transform preserve the inequality (3.1) ? In other words, why is the inverse Laplace transform of a positive function positive? Of course, this is not true in general. However, for the Laplace transform

$$L\{f(t)\}(\zeta) = \int_0^\infty e^{-\zeta t} f(t)dt,$$

the inverse Laplace transform of a positive function is positive for some classes of functions, that is, the following theorem is valid (see Theorem 2.3 in [3]).

Theorem 3.2 [3] *Let* f *be a continuous function on the interval* $[0,\infty)$ *which is of exponential order, that is, for some* $b \in R$ *it satisfies*

$$\sup_{t>0} \frac{|f(t)|}{e^{bt}} < \infty,$$

and let $F = Lf$. Then f is non-negative if and only if

$$(-1)^k F^{(k)}(s) \ge 0 \quad \textit{for all } k \ge 0 \textit{ and all } s > b. \tag{3.4}$$

In fact this positivity result implies directly from Post's inversion formula [12]

$$f(t)=\lim_{k\to\infty}\frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1}F_1^{(k)}\left(\frac{k}{t}\right) \tag{3.5}$$

for $t>0$. If (3.4) is valid then the expression on the right hand side of (3.5) is non-negative. Therefore, the limit $f(t)$ is necessarily non-negative for all t.

Lemma 3.3 *The inverse Laplace transform preserves the inequality (3.2).*

Proof of Lemma 3 .3. In our case we have

$$F(\varsigma)=F_{AB}(\varsigma)=\sum_{j=1}^{\infty}\frac{1}{|\mu_j(B)|^{p-1}(|\mu_j(B)|+\varsigma)}-\frac{1}{|\mu_j(\Omega)|^{p-1}(|\mu_j(\Omega)|+\varsigma)},\ \varsigma\geq 0.$$

To show positivity of $f(t)$ it is sufficient to check the conditions (3.4) for F_{AB}. By definition of a completely monotonic function we have

$$0\leq(-1)^{(k)}F_{AB}^{(k)}(\varsigma),k=0,1,2,...,$$

for all $\varsigma>0$, which proves the positivity of f (by Theorem (3.2)), that is, $f(t)\geq 0$ for all $t>0$. This confirms that the inverse Laplace transform preserves the inequality (3.2).

Proof of Theorem 2.1. The proof of Theorem 2.1 now follows directly from Lemma 3.1. Applying the Mellin transform

$$\frac{1}{\mu_j^l}=\frac{1}{\Gamma(l)}\int_0^\infty \exp(-\mu_j t)t^{l-1}dt,\quad \textit{for any real } l>1,$$

to the inequality (3.1) leads to

$$\sum_{j=1}^{\infty}\frac{1}{\mu_j^{q-1+l}(A)}\geq\sum_{j=1}^{\infty}\frac{1}{\mu_j^{q-1+l}(B)} \tag{3.6}$$

for any real $l>1$. Then since l is arbitrary real number >1, from (3.6) we obtain

$$\sum_{j=1}^{\infty}\frac{1}{\mu_j^{p}(A)}\geq\sum_{j=1}^{\infty}\frac{1}{\mu_j^{p}(B)} \tag{3.7}$$

for any real $p>q$. In addition, from (3.2) when $\varsigma=0$ we get that the inequality is also true when $p=q$. This completes the proof of Theorem 2.1.

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ШАТТЕН p-НОРМАСЫ ҮШІН БІР ТЕҢСІЗДІК ТУРАЛЫ

Д. Сураган

ҚР БҒМ Математика және математикалық моделдеу институты, Алматы қ.

Түйін сөздер: сингулярлық мәндер , Шаттен классы, Саймонның гипотезасы.

Аннотация. Бұл жұмыста Шаттен p-нормасы үшін абстракті теңсіздікті дәлелдедік. Бұл жұмыс Б.Саймонның гипотезасына толықтай монотондық функциялар мағынасындағы интегралдық операторлардың басыңқысын Шаттен p-нормасы үшін жауап береді.

ОБ ОДНОМ НЕРАВЕНСТВЕ Р-НОРМЫ В КЛАССЕ ШАТТЕНА

Д. Сураган

Институт математики и математического моделирования МОН РК, г. Алматы

Ключевые слова: сингулярные значения, Шаттен классы, гипотеза Саймона.

Аннотация. В статье авторы доказывают абстрактное неравенство для p-норм в классе Шаттена для компактного оператора.

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