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ITERATIVE METHODS FOR SOLVING DIFFERENCE EQUATIONS

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Abstract: In this paper some iterative methods for solving differential equations with partial derivatives are presented. In the course of solving this problem, we used the difference method and a grid pattern type cross. Numerical results obtained by the program and the results are analyzed.

The idea of the method is that fine grid problem is solved by the usual iterative method once and then from the resulting solution by converting obtain a solution on the coarse grid. And using a conventional direct method gets the error solutions on the coarse grid. Then, using the inverse transformation error of the solution to obtain a fine grid and add it to the previously obtained decision on a fine grid.

One way of solving stationary elliptic problems, including the problem, is their reduction to the decision of a fictitious non-stationary problem (hyperbolic or parabolic), the solution found that for sufficiently large values of the time it is close to the solution of the original tasks. Method is actually an iterative process of solving the problem with the conditions, and at each iteration of the unknown function values obtained by numerical solution of an auxiliary problem. In the theory of difference schemes it shows that the iterative process converges to the solution of the original problem, if there is a steady-state solution.

Introduction. For the solution of the difference of the Dirichlet problem for the Poisson equation in a rectangle the most economical are direct methods. Currently, there are routines in the algorithmic language for solving Laplace equations in a rectangle with boundary conditions of the three types, as well as with mixed boundary conditions. However, in the case when the domain is not a rectangle, or consider equations with variable coefficients, iterative methods are applied. In fact, the direct methods are economical only when the variables are separated.

Consider the theory of iterative methods for the equation $\mathrm{A}y = \varphi$, $\mathrm{A}y = \varphi$, where $\mathrm{A} = \mathrm{A}^* > 0$ Comparison of different methods was carried out for a model one-dimensional problem on the interval $0 \le x \le 1$:

$$y_{\bar{x}x} = -f(x), \;\; x = ih, \;\; 0 < i < \mathrm{N}, \quad y_0 = y_\mathrm{N} = 0.$$

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For it the operator has the form. The boundaries are determined by the operator constant

$$\delta = \frac{4}{h^2}\sin^2\frac{\pi h}{2}, \qquad \Delta = \frac{4}{h^2}\cos^2\frac{\pi h}{2}.$$

The number of iterations for the considered methods depends on the relation

$$\eta = \frac{\delta}{\Delta} = tg^2 \frac{\pi h}{2} \approx \frac{\pi^2 h^2}{4}. \qquad (1))$$

Let us now consider as a model two-dimensional Dirichlet problem in the unit square on a square $(l_1 = l_2 = 1)\,(l_1 = l_2 = 1)$ grid with a step of $h = h_1 = h_2$:

$$\mathrm{A}y = -\overset{\circ}{y}_{\bar{x}_1 x_1} - \overset{\circ}{y}_{\bar{x}_2 x_2} = \varphi, \qquad \varphi, y \in \mathrm{H}. \qquad (2)$$

The number N of intervals in each direction is equal, so that $h = 1/\mathrm{N}$.

Limits δ and Δ A operator found in attitude $\eta = \delta / \Delta$ coincides with (1). This implies that the number of iterations is not dependent on the number of measurements (if $h_1 \neq h_2$ $l_1 \neq l_2$ it is weakly dependent). Therefore, those estimates of the number of iterations of the various iterative methods that we have received for the one-dimensional model problem, valid for the two-dimensional case.

In the case of non-square grid number of iterations for the two-dimensional problem can be different from the number of iterations for the one-dimensional problem. We consider here only the alternating triangular iterative method for solving the Dirichlet problem of the difference (2).

2. **Statement of the problem**. Referring to the problem (2). The operator A is a sum $A = A_1 + A_2$

Where $A_1 y = \frac{y_{\bar{x}1}}{h_1} + \frac{y_{\bar{x}2}}{h_2}, \qquad A_2 y = -\frac{y_{x_1}}{h_1} - \frac{y_{x_2}}{h_2},$

and set $D = E$. Pairing A_1 and $A_2 : A_2 = A_1^*$ established by comparing their matrices or by using the first difference of Green's formula: $(A_1 y, \upsilon) = (y, A_1^* \upsilon) = (y, A_2 \upsilon)$.

To obtain the equation determining $\mathrm{B}y_{R+1} = (\mathrm{E} + \omega A_1)(E + \omega A_2) y_{R+1} = F_R$,

$$F_R = B\overset{\circ}{y}_R + \tau_{R+1}(\Lambda y_R + \varphi) \qquad (y_R = \mu, \qquad \overset{\circ}{y}_R = 0 \;\; x \in \gamma_h)$$

Values y_{R+1} are consistently from the equation

Hence we obtain the formula Hence we obtain the formula

$$\overset{\circ}{y}{}_R^{(1)}(i_1, i_2) = \left[\frac{\chi_1 \overset{\circ}{y}{}_R^{(1)}(i_1 - 1, i_2) + \chi_2 \overset{\circ}{y}{}_R^{(1)}(i_1, i_2 - 1) + F_R(i_1, i_2)}{(1 + \chi_1 + \chi_2)} \right],$$

$$\chi_1 = \frac{\omega}{h_1^2}, \qquad \chi_2 = \frac{\omega}{h_2^2},$$

$$\overset{\circ}{y}_{R+1}(i_1, i_2) = \left[\frac{\chi_1 \overset{\circ}{y}_{R+1}(i_1 + 1, i_2) + \chi_2 \overset{\circ}{y}_{R+1}(i_1, i_2 + 1) + \overset{\circ}{y}{}_R^{(1)}(i_1, i_2)}{(1 + \chi_1 + \chi_2)} \right]. \qquad (3)$$

To identify $\overset{\circ}{y}{}_R^{(1)}(i_1, i_2)$, select a node $i_1 = 1, \quad i_2 = 1$ in the left rectangle; Then the remaining two nodes $(i_1, i_2 - 1)$, $(i_1 - 1, i_2)$ and $\{(i_1, i_2), (i_1 - 1, i_2), (i_1, i_2 - 1)\}$ template are on the boundary and, therefore $\overset{\circ}{y}{}^{(1)}(i_1 - 1, i_2) = \overset{\circ}{y}{}^{(1)}(i_1, i_2 - 1) = 0$ known. Knowing $\overset{\circ}{y}{}_R^{(1)}$ when $i_1 = 1, \quad i_2 = 1$ successively $\overset{\circ}{y}{}_R^{(1)}$ find $i_1 = 2, 3, \ldots, \mathrm{N}_1 - 1$ and $i_2 = 1$ when (the first line). Next $i_2 = 2$, think and find $\overset{\circ}{y}{}_R^{(1)}$ the series on the second line at $i_1 = 1, 2, \ldots, N - 1$. To determine to $\overset{\circ}{y}_{R+1}$ perform calculations

$\{(i_1,\ i_2),\quad (i_1+1,\ i_2),\quad (i_1, i_2+1)\}$ on the template in columns from top to bottom: fix $i_1 = N_1-1,\ \ N_1-2,\ldots,2,1,$ and at each i_1 change $i_2 = N_2-1,\ \ N_2-2,\ldots,2,1.$ Starts counting $\overset{\circ}{y}_{R+1}$ from the node $(i_1 = N_1-1,\ \ i_2 = N_2-1)$ in the upper right corner. It should be $\overset{\circ}{y}_{R+1}$ noted that the account can also be conducted in rows from right to left :fix $i_2 = N_2-1,\ \ N_2-2,\ldots,2,1$ and at each i_2 change $i_1 = N_1-1,\ \ N_1-2,\ldots,2,1.$ However, the calculation $\overset{\circ}{y}{}_R^{(1)}$ can be carried out not by rows and columns upwards. It can be seen from the formulas themselves.

The calculations are carried out by the recurrence formulas (11); Account apparently stable. An algorithm of this type, as already noted, an algorithm called the running computation.

We count the number of arithmetic operations per grid point: calculation requires 10 additions and 10 multiplications; for a given computation requires 4 operations of addition and multiplication operations 6. Total required for the determination of a single node to spend 14 additions and 16 multiplications. The number of operations can be reduced when storing in memory not just one but two sequences $\{y_R\}$ and $\{w_{R+1}\}$ for determining y_{R+1} the use of the algorithm

$$(\mathrm{E}+\omega\, A_1)\overset{\circ}{w}_{R+1/2} = \Lambda y_R + f, \qquad (\mathrm{E}+\omega\, A_2)\overset{\circ}{w}_{R+1} = \overset{\circ}{w}_{R+1/2},$$

$$y_{R+1} = y_R + \tau_{R+1}\overset{\circ}{w}_{R+1}.$$

In this case, the transition y_R from y_{R+1} a fairly 10 additions and 10 multiplications per node.

For example, Find a continuous function u (x, y) satisfying the rectangular area $\Omega = \{(x,y)| 0\le x\le a, 0\le y\le b\}$ within the Laplace equation $\Delta u \equiv \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$ and taking on the boundary of the set values Ω, $u(0,y) = f_1(y),\ y\in[0,b]$, $u(a,y) = f_2(y),\ y\in[0,b]$,

$u(x,0) = f_3(x),\ x\in[0,a]$, $u(x,b) = f_4(x),\ x\in[0,a]$,

where f_1, f_2, f_3, f_4 - given functions,

$f_1 = y, f_2 = y+e, f_3 = e^x, f_4 = e^{x+1}$. We assume that u (x, y) is continuous on the boundary Ω, etc. $f_1(0) = f_3(0)$, $f_1(b) = f_4(0)$, $f_2(0) = f_3(a)$, $f_2(b) = f_4(a)$. Choosing steps h, l x and y respectively, we construct a grid $x_i = ih$, $i = 0,1,\ldots,n$, $y_j = jl$, $j = 0,1,\ldots,m$ where $x_n = nh = a$, $y_m = ml = b$. Introducing the notation $u_{ij} = u(x_i, y_j)$, we approximate $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ the partial derivatives in each internal node of the grid central second order

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + O(h^2)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{l^2} + O(l^2)$$

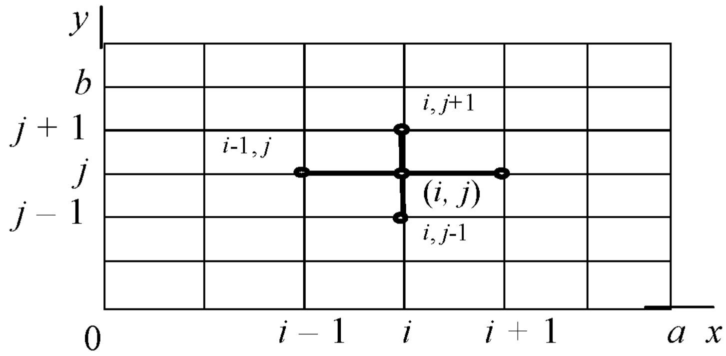
and replace the Laplace equation finite-difference equation

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{l^2} = 0 \qquad (4)$$

$i = 1,\ldots,n-1$ $j = 1,\ldots,m-1$ Accuracy replacement of the differential equation is of the difference $O(h^2)+O(l^2)$. Equation (4) together with the values in the boundary nodes form a system of linear algebraic equations relatively to approximate u_{ij} values of u (x, y) at the grid points (x_i, y_j). The simplest form of this system is at: $u_{ij} = (u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})/4$ (5)

$$u_{i0} = f_3(x_i),\ u_{im} = f_4(x_i),\ u_{0j} = f_1(y_j),\ u_{nj} = f_2(y_j),$$
$$i = 1,...,n-1,\ j = 1,...,m-1.$$

Upon receipt of the grid equations (5) was used circuit nodes shown in Fig. 1. The set of nodes used to approximate equation at a point called the template. This paper uses the template of the "cross".



Numerical solution of the Dirichlet problem for the Laplace equation in a rectangle is to find approximate values u_{ij} of the unknown function u (x, y) in the internal nodes of the grid. To determine u_{ij} the quantities required to solve a system of linear algebraic equations (5).

In this paper, it is solved by the Gauss-Seidel method, which consists in building a sequence of iterations of the form $u_{ij}^{(s+1)} = \frac{1}{4}(u_{i-1,j}^{(s+1)} + u_{i+1,j}^{(s)} + u_{i,j+1}^{(s)} + u_{i,j-1}^{(s+1)})$ .If $s \to \infty$ the sequence $u_{ij}^{(s)}$ converges to the exact solution of the system (5). As a condition to the end of the iteration process can take

$$\max\left|u_{ij}^{(s)} - u_{ij}^{(s+1)}\right| < \varepsilon,\ 1 \le i \le n-1,\ 1 \le j \le m-1.$$

Conclusion. Thus, the error of the approximate solutions obtained by the grid method, consists of two errors: errors of approximation of differential equations by difference; error arising from the approximate solution of the system of difference equations (5). It is known that the difference scheme described here has the property of stability and convergence. Stability of the scheme means that small changes in initial conditions lead to small changes in the solution of the difference problem. Only such schemes should be applied in the actual calculations. Convergence of the scheme means that as the grid spacing to zero ($h \to 0$), the solution of the difference problem tends in some sense to the solution of the original problem. Thus, by choosing a sufficiently small step h, can be arbitrarily accurately solve the original problem.

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ИТЕРАЦИОННЫЕ МЕТОДЫ РЕШЕНИЯ СЕТОЧНЫХ УРАВНЕНИЙ

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Ключевые слова: решение, итерационный метод, узел, сетка.

Аннотация. В этой статье некоторые итерационные методы для решения дифференциальных уравнений с частными производными. В ходе решения этой проблемы мы использовали метод разницы и крест сетки типа рисунок. Анализируются численные результаты, полученные с помощью программы.

Идея метода состоит в том, что проблема решается обычным итерационным методом один раз и затем из полученного решения путем преобразования получить решение на грубой сетке. И с помощью обычного метода прямого получает решения об ошибках на грубой сетке. Затем, используя обратную ошибку преобразования решения, чтобы получить мелкую сетку и добавить его к полученному ранее решения на мелкой сетке. Одним из способов решения стационарных эллиптических задач, в том числе и краевой задачи, является их сведение к решению некоторой фиктивной нестационарной задачи (гиперболической или параболической), найденное решение которой при достаточно больших значениях времени t близко к решению исходной задачи. Такой способ решения называется методом установления. Метод установления фактически представляет итерационный процесс решения задачи с условиями, причем на каждой итерации значения искомой функции получаются путем численного решения некоторой вспомогательной задачи. В теории разностных схем показано, что этот итерационный процесс сходится к решению исходной задачи, если такое стационарное решение существует.

ТОР ТЕҢДЕУЛЕРІНІ ИТЕРАЦИЯЛЫҚ ӘДІСПЕН ШЫҒАРУ

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Түйін сөздер: шешім, итерациялық қадамдар, тор, түйін.

Аннотация. Бұл мақалада дербес туындылы дифференциалды теңдеулерді шешудің итерациялық әдістері қарастырылған. Есепті шешу барысында біз айырым мен крест-торын қолдандық. Компьютерлік программамен алынған нәтижелер сараланды.

Бұл әдісте тордағы мәселе қарапайым болып шешіледі де, оны күрделі торға ауыстырады. Одан кейін карапайым тормен күрделі торды бір-бірімен жалғайды.

Бұл әдіс шектік есептерді шығару барысында көптеп қолданылады. Берілген есепті қарапайым есептер түріне келтіріп алып, содан кейін жіберілген қателіктерді ескере отрып берілген есептің шешіміне жақындатады. Бұл әдіс әлбетте бірнеше итерацияны ұйымдастырумен жүзеге асады. Әрбір итерациядағы алынған шешімге алдыңғы қадамдағы шешім әсер етіп отырады. Алынған шешімдердің жинақтылығын дәлелдей білу керек.

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