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ANALYTICAL SOLUTION OF THE HEAT EQUATION WITH DISCONTINUOUS COEFFICIENTS

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Abstract. The main idea of the developed method is based on the use of Integral Error Functions and Heat polynomials which a priori satisfy the heat equation. Linear combination of these functions allows to solve heat equations in domains with moving boundaries with discontinuous coefficients.

Introduction. Solution of heat equation by heat polynomials in domains with nonmoving boundaries were considered in [1]. Heat distribution in degenerate domains substantially complicates the problem and cannot be solved by classical methods represented in [2] and [3]. Only for special cases it is possible to construct Heat potentials, however in general solution reduced to the system of Integro-differential equations which contains singularities. In this paper we consider analytical solution of the heat equation with discontinuous coefficients in domain with moving boundary which degenerate at the initial time. This problem is a part of study where the two phase spherical Stefan problem with two moving boundaries has to be solved. We follow the method represented in [4] and utilize Heat polynomials to solve the problem.

Problem statement. It is required to find the solution of the Heat Equation

$$\frac{\partial u_1}{\partial t} = a_1^2 \frac{\partial^2 u_1}{\partial x^2}, \quad 0 < x < \beta(t), \quad t > 0 \quad (1),$$

$$\frac{\partial u_2}{\partial t} = a_2^2 \frac{\partial^2 u_2}{\partial x^2}, \quad \beta(t) < x < \infty, \quad t > 0 \quad (2),$$

with following boundary conditions:

$$x=0: \quad -\lambda_1 \left[b \frac{\partial u_1}{\partial x} - u_1 \right]_{x=0} = b^2 P(t), \quad (3),$$

$$u_2(\infty, t) = 0 \quad (4),$$

$$x=\beta(t): \quad u_1(\beta(t), t) = 0 \quad (5),$$

$$u_2(\beta(t), t) = 0 \quad (6).$$

Problem solution. We represent solution in the following form:

$$u_1(x, t) = \sum_{n=0}^{\infty} A_{2n} (2a_1 t)^n \left[i^{2n} \operatorname{erfc} \frac{-x}{2a_1 \sqrt{t}} + i^{2n} \operatorname{erfc} \frac{x}{2a_1 \sqrt{t}} \right] + \\ + \sum_{n=0}^{\infty} A_{2n+1} (2a_1 t)^{\frac{2n+1}{2}} \left[i^{2n+1} \operatorname{erfc} \frac{-x}{2a_1 \sqrt{t}} - i^{2n+1} \operatorname{erfc} \frac{x}{2a_1 \sqrt{t}} \right] \quad (7),$$

$$u_2(x,t) = \sum_{n=0}^{\infty} C_n (2a_2 t)^{\frac{n}{2}} i^n erfc \frac{x}{2a_2 \sqrt{t}} \quad (8)$$

Where coefficients A_{2n}, A_{2n+1}, C_n have to be found. Moreover, it is necessary to find unknown moving $\beta(t)$. Using Hermite polynomials we represent (7) in the form of Heat polynomials:

$$u_1(x,t) = \sum_{n=0}^{\infty} A_{2n} \sum_{m=0}^n x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^n x^{2n-2m+1} t^m \beta_{2n+1,m} \quad (9)$$

Making substitution $\sqrt{t} = \tau$

$$u_1(x,t) = \sum_{n=0}^{\infty} A_{2n} \sum_{m=0}^n x^{2n-2m} \tau^{2m} \beta_{2n,m} + A_{2n+1} \sum_{m=0}^n x^{2n-2m+1} \tau^{2m} \beta_{2n+1,m} \quad (10)$$

From (5) for $x = 0$

$$(b \frac{\partial u_1}{\partial x} - u_1)_{x=0} = \frac{b^2}{-\lambda_1} P(t)$$

Using above expression we have

$$\begin{aligned} b \sum_{n=0}^{\infty} A_{2n+1} t^n \beta_{2n+1,n} - \sum_{n=0}^{\infty} A_{2n} t^n \beta_{2n,n} &= \frac{b^2}{-\alpha_1} P(t) \\ b A_{2n+1} \beta_{2n+1} - A_{2n} \beta_{2n,n} &= \frac{b^2}{-\alpha_1} \cdot \frac{P^{(n)}(0)}{n!} \\ A_{2n+1} &= A_{2n} \frac{\beta_{2n,n}}{b \cdot \beta_{2n+1,n}} - \frac{b}{\alpha_1 \beta_{2n+1,n}} \cdot \frac{P^{(n)}(0)}{n!} \end{aligned} \quad (11)$$

To find A_{2n} we use multinomial coefficients of Newton's Polynomials.

It is known that

$$(x_1 + x_2 + \dots + x_m)^n = \sum_{k_1+k_2+\dots+k_m=n} \binom{n}{k_1, k_2, \dots, k_m} \prod_{1 \leq t \leq m} x_t^{k_t}$$

where $\binom{n}{k_1, k_2, k_3, \dots, k_m} = \frac{n!}{k_1! k_2! \dots k_m!}$ is a *multinomial coefficient*

$$\text{for } \beta(t) = \alpha_1 t^{1/2} + \alpha_2 t^1 + \alpha_{n+1} t^{3/2} + \dots = \sum_{n=0}^{\infty} \alpha_{n+1} t^{\frac{n+1}{2}} \quad (12)$$

after making substitution $\tau = \sqrt{t}$ we have

$$(\alpha_1 \tau + \alpha_2 \tau^2 + \dots + \alpha_{m+1} \tau^{m+1})^n = \sum_{k_1+k_2+\dots+k_{m+1}=n} \binom{n}{k_1, k_2, \dots, k_{m+1}} \cdot \alpha_1^{k_1} \alpha_2^{k_2} \dots \alpha_{m+1}^{k_{m+1}} t^{k_1+2k_2+\dots+(m+1)k_{m+1}} \quad (13)$$

where

$$\binom{n}{k_1, k_2, \dots, k_{m+1}} \alpha_1^{k_1} \alpha_2^{k_2} \dots \alpha_{m+1}^{k_{m+1}} \quad (14)$$

is a multinomial coefficient in our case

Thus to derive recurrent formula for A_{2n} , we take both sides of (5), $2k$ – times derivatives at $\tau = 0$, we use multinomial coefficients and get following expressions.

$$0 \equiv 0^{(4l)} = \sum_{n=1}^l A_{2n} \sum_{m=0}^{n-1} C_{2n,m} [4l] + \sum_{n=l+1}^{2l-1} A_{2n} \sum_{m=0}^{2l-n-1} C_{2n,m+2(n-l)} [4l] \beta_{2n,m+2(n-l)} + A_{4l} \beta_{4l,2l} + \\ + \sum_{n=1}^l A_{2n-1} \sum_{m=0}^{n-1} C_{2n-1,m} [4l] \beta_{2n-1,m} + \sum_{n=l+1}^{2l} A_{2n-1} \sum_{m=0}^{2l-n} C_{2n-1,m+2(n-l)-1} [4l] \beta_{2n-1,m+2(n-l)-1}$$

where $l=1,2,\dots$ and

$$0 \equiv 0^{(2(2l-1))} = \sum_{n=1}^{l-1} A_{2n} \sum_{m=0}^{n-1} C_{2n,m} [2(2l-1)] \beta_{2n,m} + \sum_{n=l}^{2l-2} A_{2n} \sum_{m=1}^{2l-n-1} C_{2n,m+2(n-l)} [2(2l-1)] \beta_{2n,m+2(n-l)} + \\ + A_{4l-2} \beta_{4l-2,2l-1} [2(2l-1)] + \sum_{n=0}^{l-1} A_{2n+1} \sum_{m=0}^n C_{2n+1,m} [2(2l-1)] \beta_{2n+1,m} + \\ + \sum_{n=l}^{2l-2} A_{2n+1} \sum_{m=1}^{2l-n-1} C_{2n+1,m+2(n-l)+1} [2(2l-1)] \beta_{2n+1,m+2(n-l)+1}$$

(16)

$$A_0 = 0$$

Thus A_{2n} , coefficients are found explicitly and can be calculated from (15) and (16) where $C_{i,j}[4l]$ or $C_{i,j}[4l-2]$ multinomial coefficients or sums of coefficients at $\beta_{i,j}$.

To calculate C_n we apply Leibniz, Faa Di Bruno's formulas and Bell polynomials

Using Leibniz formula we have

$$\left. \frac{\partial^k [2^{n/2} \tau^n \cdot i^n \operatorname{erfc} \beta]}{\partial \tau^k} \right|_{\tau=0} = \begin{cases} 0, & \text{for } k < n \\ \frac{2^{n/2} k!}{(k-n)!} \cdot [i^n \operatorname{erfc} \beta]^{(k-n)}, & \text{for } k \geq n \end{cases} \quad (17)$$

Using Faa Di Bruno's formula and Bell polynomials for a derivative of a composite function we have

$$\left. \frac{\partial^{k-n} [i^n \operatorname{erfc}(\pm \beta)]}{\partial \tau^{k-n}} \right|_{\tau=0} = \sum_{m=1}^{k-n} (i^n \operatorname{erfc}(\pm \beta))^{(m)} \Big|_{\beta=0} \cdot B_{k-n,m} (\beta'(\tau), \beta''(\tau), \dots, \beta^{(k-n-m+1)}(\tau)) \Big|_{\tau=0} \quad (18)$$

$$\text{where } B_{k-n,m} = \sum \frac{(k-n)!}{j_1! j_2! \dots j_{k-n-m+1}!} \cdot \beta_1^{j_1} \beta_2^{j_2} \beta_3^{j_3} \dots \beta_{k-n-m+1}^{j_{k-n-m+1}} \quad (19)$$

and j_1, j_2, \dots satisfy following equations

$$\begin{aligned} j_1 + j_2 + \dots + j_{k-n-m+1} &= m \\ j_1 + 2j_2 + \dots + (k-n-m+1)j_{k-n-m+1} &= k-n \end{aligned}$$

for $k \geq n$

$$[i^n \operatorname{erfc}(\pm \beta)]^{(m)} \Big|_{\beta=0} = (-1)^m i^{n-m} \operatorname{erfc} 0 = (\mp 1)^m \frac{\Gamma(\frac{n-m+1}{2})}{(n-m)! \sqrt{\pi}} \quad (20)$$

From $x = \beta(\tau)$ we have

$$\sum_{n=0}^{k-1} C_n \cdot \mu + C_k \cdot \mu = 0 \quad (21)$$

where

$$\mu = (2)^{\frac{k}{2}} \frac{k!}{(k-n)!} \sum_{m=1}^{k-n} (-1)^m \frac{\Gamma(\frac{n-m+1}{2})}{(n-m)! \sqrt{\pi}} \cdot \sum \frac{(k-n)!}{j_1! j_2! \dots j_{k-n-m+1}!} \cdot \beta_1^{j_1} \beta_2^{j_2} \beta_3^{j_3} \dots \beta_{k-n-m+1}^{j_{k-n-m+1}}$$

Thus utilizing (11), (15), (16) and (21) we find A_{2n}, A_{2n+1}, C_n coefficients of functions (7) and (8). Convergence of (7) and (8) can be proved by following the analogy of the proof represented in [4].

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ЖЫЛУӨТКІЗГІШТІК ТЕНДЕУІНІҚ ЖЫЛЖЫМАЛЫ АЙМАҚТАРДА АНАЛИТИКАЛЫҚ ШЕШІМІ

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Тірек сөздер: интегралды Қателіктер Функциясы, жылу полиномдары, жылжымалы аймактар.

Аннотация. Жылуөткізгіштік тендеуінің интегралды қателіктер функциялары және жылу полиномдары арқылы аналитикалық шешими табылған.

АНАЛИТИЧЕСКОЕ РЕШЕНИЕ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ С РАЗРЫВНЫМИ КОЭФФИЦИЕНТАМИ

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Ключевые слова: интегральная Функция Ошибок, тепловые полиномы, подвижные границы.

Аннотация. Найдено аналитическое решение уравнения теплопроводности с разрывными коэффициентами методом интегральных функций ошибок и тепловых полиномов.

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REPRESENTATIONS OF S_n ON SOME ROOTED TREES IN FREE RIGHT-COMMUTATIVE ALGEBRA

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Key words: free algebra, multi-linear part, identity, irreducible module, basis, rooted tree, Young symmetrizer, group of automorphisms, cycle index, permutation module.

Abstract. Algebra with identity $(a \cdot b) \cdot c = (a \cdot c) \cdot b$ is called a right-commutative. In [2] the basis of the free right-commutative algebra constructed by rooted trees. Studies varieties of free algebras lead to the study of