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ANALYTICAL SOLUTION OF ONE PHASE STEFAN PROBLEM BY HEAT POLYNOMIALS

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Abstract. Solution of One Phase Stefan Problem with degenerate free boundary represented in explicit analytic form. The developed method is based on the use of Integral Error Functions and its properties. The main idea of this method is to find coefficients of linear combination of Integral Error Functions which a priori satisfy the heat equation. Convergence of series proved. Method was tested and applied in experimental problem.

1. Introduction. A wide range of transient phenomena in the field of heat and mass transfer, low-temperature plasma, filtration and other evolutionary processes associated with phase transformation of materials considered in quite extensive literature; see, e.g., [1-7], and a long bibliography on these problems given in [8] leads to the necessity of solving boundary value problems of heat and mass transfer with free moving inter-phase boundaries.

From theoretical point of view, these problems are among the most challenging problems in the theory of non-linear parabolic equations, which along with the desired solution, an unknown moving boundary has to be found (Stefan type problems). In some specific cases it is possible to construct Heat potentials for which, boundary value problems can be reduced to integral equations [4], [5], [9]. However, in the case of domains that are degenerate at the initial time, there are additional difficulties because of the singularity of the integral equations, which belong to the class of pseudo - Volterra equations which are unsolvable in the general case, by the method of successive approximations [9]. Therefore, investigation of methods for the solution of Stefan type problems is an actual mathematical problem.

Tracking answers of these questions will be as following. In the first section introductory information and some properties of Integral Error Functions necessary for elaborating different methods (e.g. Heat Polynomials method) are represented. In the second section one phase Stefan problem stated and the solution represented. In the third section test problem with given exact solution is solved by proposed method. Fourth section is devoted to discussion of further development of method for wider class of problems.

1.1. Integral Error Functions

The integral error functions determined by recurrent formulas

$$i^n erfcx = \int_x^\infty i^{n-1} erfcvdv, \quad \text{n=1,2,...} \quad i^0 erfcx \equiv erfcx = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-v^2)dv \tag{1}$$

where

$$erfx = 1 - erfcx = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-v^2)dv \tag{2}$$

It is well known that the Integral Error Functions

$$u_n(\pm x,t)=t^{\frac{n}{2}}i^n erfc\frac{\pm x}{2a\sqrt{t}} \tag{3}$$

exactly satisfy the heat equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2} \tag{4}$$

and by superposition principle, linear combination of (6) or even series also satisfy (4)

$$u(x,t)=\sum_{n=0}^{\infty}[A_n u_n(x,t)+B_n u_n(-x,t)] \tag{5}$$

We consider (4) and solution (5) in degenerate domain where constants A_n, B_n have to be determined and can be derived by substituting (5) into boundary conditions at $x=0$ and $x=\alpha(t)$, if given boundary functions can be expanded into Taylor series with powers t or $\sqrt{t}$.

1.2. Properties of Integral Error Functions

1. Using formula for Hermite polynomials one can derive

$$\mathrm{i^n erfc(-x)+(-1)^n i^n erfcx}=\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{x^{n-2m}}{2^{2m-1}m!(n-2m)!} \tag{6}$$

and represent (5) in the form of heat polynomials

$$u(x,t)=\sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\right] \tag{7}$$

where

$$\beta_{n,m}=\frac{1}{2^{n+m-1}\cdot m!\cdot(n-2m)!} \tag{8}$$

2. Using L'Hopital rule it is not difficult to show that

$$\lim_{x\to\infty}\frac{i^n erfc(-x)}{x^n}=\frac{2}{n!} \tag{9}$$

2. One Phase Stefan Problem and its solution

Definition of One phase stefan problem, it's physical interpretation.

2.1. Problem statement

Solve the Heat Equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\quad 0<x<\alpha(t),\quad 0<t<\infty, \tag{10}$$

Subject to

$$-\lambda\left(\frac{\partial u}{\partial x}-u\right)\bigg|_{x=0}=P(t),\quad 0<t<\infty, \tag{11}$$

$$u_{x=\alpha(t)}=U_m,\quad 0<t<\infty, \tag{12}$$

$$-\lambda\left(\frac{\partial u}{\partial x}-u\right)\bigg|_{x=\alpha(t)}=\frac{d\alpha(t)}{dt},\quad 0<x<\alpha(t) \tag{13}$$

$$u(0,0)=0 \tag{14}$$

It is necessary to find *u(x,t)*-heat transfer function and free moving boundary $\alpha(t)$ which can be represented in the following form

$$\alpha(t)=\sum_{n=1}^{\infty}\alpha_n t^{\frac{n}{2}} \tag{15}$$

2.2. Solution:

We represent solution in the following form

$$u(x,t)=\sum_{n=0}^{\infty}A_{2n}(2at)^n\left[i^{2n}\operatorname{erfc}\frac{(-x)}{2\sqrt{t}}+i^{2n+1}\operatorname{erfc}\frac{x}{2\sqrt{t}}\right]+\sum_{n=0}^{\infty}A_{2n+1}(2at)^{\frac{2n+1}{2}}\left[i^{2n+1}\operatorname{erfc}\frac{(-x)}{2\sqrt{t}}-i^{2n+1}\operatorname{erfc}\frac{x}{2\sqrt{t}}\right] \tag{16}$$

and by property (6) we represent (16) in the form of Heat Polynomials

$$u(x,t)=\sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\right] \tag{17}$$

where A_{2n} and A_{2n+1} coefficients have to be found.

From (11)

$$A_{2n}+A_{2n+1}=\frac{P_n}{\beta_{2n+1,n}} \tag{18}$$

where

$$P_n=\frac{P^{(n)}(0)}{n!} \tag{19}$$

Making substitution $\sqrt{t}=\tau$ in (17) we have

$$u(x,t)=\sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n}x^{2n-2m}\tau^{2m}\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}\tau^{2m}\beta_{2n+1,m}\right] \tag{20}$$

we take both sides of (12) 2k – times derivatives at $\tau=0$ and get following expression

$$0\equiv 0^{(2k)}=\sum_{n=1}^{k-1}A_{2n}\sum_{m=0}^{n}c[2k]\beta_{2n,m}+A_{2k}\sum_{m=0}^{n}c[2k]\beta_{2n,m}+\sum_{n=1}^{k}A_{2n-1}\sum_{m=0}^{n}c[2k]\beta_{2n-1,m} \tag{21}$$

It is easy to see that, if we express from (18) A_{2n+1} in terms of A_{2n} and substitute it into (21) we obtain recurrent formula for A_{2n}

Thus A_{2n} coefficients are found from (21) where $c[2k]$ multinomial coefficients, which can be found as following are:

From (13)

for 2k-th derivative we consider (15) in the form $\alpha(t)=\sum_{n=1}^{k}\alpha_n t^{\frac{n}{2}}$ because other coefficients $\alpha_i, i>2k$ don't affect C[2k]

$$\left(\alpha_1\tau+\alpha_2\tau^2+\ldots+\alpha_2\tau^{2k}\right)^n=\sum_{j_1+j_2+\ldots+j_{2k}=n}\binom{n}{j_1,j_2,\ldots,j_{2k}}\alpha_1^{j_1}\alpha_2^{j_2}\ldots\alpha_{2k}^{j_{2k}}\tau^{j_1+2j_2+\ldots+(2k)j_{2k}}$$

We have $[(\alpha(\tau))^n]_{\tau=0}^{(2k)}=[(\alpha_1\tau+\alpha_2\tau^2+\ldots+\alpha_2\tau^{2k})^n]_{\tau=0}^{(2k)}=c[2k]$

where

$$c[2k]=\sum_{j_1+j_2+\ldots+j_{2k}=n}\binom{n}{j_1,j_2,\ldots,j_{2k}}\alpha_1^{j_1}\alpha_2^{j_2}\ldots\alpha_{2k}^{j_{2k}} \tag{22}$$

and j_i, $i=1,\ldots,2k$ satisfy the equations

$$j_1 + j_2 + ... + j_{2k} = n \tag{23}$$

$$j_1 + 2j_2 + ... + (2k)j_{2k} = 2k \tag{24}$$

Making substitution $\sqrt{t} = \tau$ and taking both sides of (18) 2k and (2k+1) – times derivatives at $\tau = 0$ it is possible to derive α_n coefficients as following

$$[(\alpha(\tau))^n]^{(2k)}_{\tau=0} \equiv \alpha_{2k+1} = \sum_{n=1}^{k} A_{2n} \sum_{m=0}^{n-1} [2(n-m)]c[2k]\beta_{2n,m} + \sum_{n=1}^{k-1} A_{2n+1} \sum_{m=0}^{n-1} [2(n-m)-1]c[2k]\beta_{2n+1,m} \tag{25}$$

$$[(\alpha(\tau))^n]^{(2k+1)}_{\tau=0} \equiv \alpha_{2k+2} = \sum_{n=1}^{k} A_{2n} \sum_{m=0}^{n-1} [2(n-m)]c[2k+1]\beta_{2n,m} + \sum_{n=1}^{k} A_{2n+1} \sum_{m=0}^{n-1} [2(n-m)+1]c[2k]\beta_{2n+1,m} \tag{26}$$

for k=1,2,…

Thus from (18),(21),(25),(26) we derive A_{2n}, A_{2n+1} and α_{2k+1}, α_{2k+2} respectively where n=1,2,…

2.3. Convergence

Let $\alpha(t_0) = \mu_0$ for any time $t = t_0$. Then the series

$$u(x,t_0) = \sum_{n=0}^{\infty} A_{2n}(2at_0)^n \left[i^{2n}erfc\frac{(-\mu_0)}{2\sqrt{t_0}} + i^{2n}erfc\frac{\mu_0}{2\sqrt{t_0}} \right] + $$
$$+ \sum_{n=0}^{\infty} A_{2n+1}(2at_0)^{\frac{2n+1}{2}} \left[i^{2n+1}erfc\frac{(-\mu_0)}{2\sqrt{t_0}} - i^{2n+1}erfc\frac{\mu_0}{2\sqrt{t_0}} \right]$$

should be convergent, because $U = U_m$ on the interface. Therefore there exist a constant C independent of *n*, such that

$$|A_{2n+1}| < C/(2at_0)^{\frac{2n+1}{2}} \left[i^{2n+1}erfc\frac{(-\mu_0)}{2\sqrt{t_0}} + i^{2n+1}erfc\frac{\mu_0}{2\sqrt{t_0}} \right] \tag{27}$$

Multiplying both sides of (27) by $(2at)^{\frac{2n+1}{2}} \left[i^{2n+1}erfc\frac{(-\alpha(t))}{2\sqrt{t}} + i^{2n+1}erfc\frac{\alpha(t)}{2\sqrt{t}} \right]$ and taking sum we obtain

$$\sum_{n=0}^{\infty} A_{2n+1}(2at)^{\frac{2n+1}{2}} \left[i^{2n+1}erfc\frac{(-\alpha(t))}{2\sqrt{t}} + i^{2n+1}erfc\frac{\alpha(t)}{2\sqrt{t}} \right] <$$
$$< C\frac{\sum_{n=0}^{\infty} A_{2n+1}(2at)^{\frac{2n+1}{2}} \left[i^{2n+1}erfc\frac{(-\alpha(t))}{2\sqrt{t}} + i^{2n+1}erfc\frac{\alpha(t)}{2\sqrt{t}} \right]}{\sum_{n=0}^{\infty} A_{2n+1}(2at_0)^{\frac{2n+1}{2}} \left[i^{2n+1}erfc\frac{(-\mu_0)}{2\sqrt{t}} + i^{2n+1}erfc\frac{\mu_0}{2\sqrt{t}} \right]} < C\sum_{n=0}^{\infty}\left(\frac{t}{t_0}\right)^{\frac{2n+1}{2}} \tag{28}$$

In the same manner, similar estimations for α_n can be obtained from (25) and (26)

Thus on the base of monotonicity of function $\left[i^{2n+1}erfc\frac{(-\alpha(t))}{2\sqrt{t}} + i^{2n+1}erfc\frac{\alpha(t)}{2\sqrt{t}} \right]$, and above estimations it is possible to conclude that series (17) is convergent.

3. Test problem

Solve the Heat Equation

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \alpha(t), \quad 0 < t < 1, \tag{29}$$

Subject to

$$-\left.\frac{\partial u}{\partial x}\right|_{x=0} = e^t, \quad 0<t<1, \tag{30}$$

$$u_{x=\alpha(t)} = 0, \quad 0<t<1, \tag{31}$$

$$-\left.\frac{\partial u}{\partial x}\right|_{x=\alpha(t)} = \frac{\partial \alpha(t)}{\partial t}, \quad 0<x<\alpha(t) \tag{32}$$

$$u(0,0)=0 \tag{33}$$

Exact solution:

$$u(x,t)=e^{t-x}-1 \text{ and } \alpha(t)=t \tag{34}$$

Solution by Heat polynomials:

We consider solution in the form of heat polynomials (20) and it is necessary to determine A_{2n}, A_{2n+1} coefficients and free boundary (15). It is easy to see that from (30) we can directly derive A_{2n+1} coefficients

$$A_{2n+1} = \frac{P_n}{\beta_{2n+1,n}} \tag{35}$$

where P(t)=e^t, and $P_n = \frac{P^{(n)}(0)}{n!}$.

In the same manner, to derive recurrent formula for A_{2n}, we take both sides of (31) 2k – times derivatives at $\tau = 0$ and get following expression

$$0 \equiv 0^{(2k)} = \sum_{n=1}^{k-1} A_{2n} \sum_{m=0}^{n} c[2k]\beta_{2n,m} + A_{2k} \sum_{m=0}^{n} c[2k]\beta_{2n,m} + \sum_{n=1}^{k} A_{2n-1} \sum_{m=0}^{n} c[2k]\beta_{2n-1,m} \tag{36}$$

Making substitution $\sqrt{t} = \tau$ and taking both sides of (32) 2k and (2k+1) – times derivatives at $\tau = 0$ we derive recurrent formulas for α_n coefficients (25), (26) for k=1,2,…

For example if we expand (20) and calculate first four coefficients and α_1, α_2 we can easily find that $A_0 = 0, A_2 = 0, A_1 = -1/2, A_3 = -8$ and $\alpha_1 = 1, \alpha_2 = 0$ which agree with exact solution (34).

4. Discussion and conclusion

Successful applications of represented method induce the question:

1) Is it possible to elaborate similar methods for the Stefan type problems with several phases and for two or three dimensional cases?

2) It can be shown that Generalized Integral Functions and their linear combinations satisfy the equation

$$\frac{\partial \theta}{\partial t} = a^2 \left(\frac{\partial^2 \theta}{\partial z^2} + \frac{\nu}{z}\frac{\partial \theta}{\partial z}\right)$$

where $\nu = 1,2,3$ are plane, cylindrical and spherical cases, which is very important for the modeling of Heat transfer in solid with variable cross section.

3) One of the objectives of further research is to prove that mentioned Generalized Integral Functions or elaborated Degenerate Hypergeometric Functions and their linear combinations satisfy above equation for any ν.

Main results namely coefficients of function (17) A_{2k}, A_{2k+1} and free boundary are found analytically, convergence of (17) is proved.

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БІР ФАЗАЛЫ СТЕФАН ЕСЕБІНІҢ ЖЫЛУ ПОЛИНОМДАРЫ АРҚЫЛЫ АНАЛИТИКАЛЫҚ ШЕШІМІ

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Тірек сөздер: бір фазалы Стефан, интегралды функциясының кемшілігі, жылуөткізгіштік теңдеуі, эксперименттік мәселелер.

Аннотация. Бастапқы уақытта құлдырайтын, жылжымалы шекарасы бар аймақтарда бір фазалы Стефан есебінің жылу полиномдар арқылы аналитикалық шешімі табылған.

АНАЛИТИЧЕСКОЕ РЕШЕНИЕ ОДНОФАЗНОЙ ЗАДАЧИ СТЕФАНА МЕТОДОМ ТЕПЛОВЫХ ПОЛИНОМОВ

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Ключевые слова: однофазная задача Стефанаса, интегральная погрешность функции, уравнения теплопроводности, экспериментальные проблемы.

Аннотация. Найдено аналитическое решение однофазной задачи Стефанас вырождающейся в начальный момент времени границей методом тепловых полиномов.

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