N E W S
OF THE NATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
PHYSICO-MATHEMATICAL SERIES
ISSN 1991-346X
Volume 5, Number 309 (2016), 160 – 167

UDK 517.927

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NUMERICAL RESULTS OF SPECIAL GRID OF STRINGS

Abstract. In this paper some numerical results about natural oscillations of a special grid of strings are considered. The problem of a string of natural frequencies of such grid is reduced to the Sturm-Liouville problem on the grid. The main result of the work is given in programming language Matlab. The finite-difference method is used to study the problem of special grid of string.
Keywords: string, vertex, edges, node, eigenvalue, eigenvector.

Introduction. In this paper we give some numerical results about natural oscillations of a special grid of strings (see [1], [2]). The system consists of a finite number of strings attached to each other as it is shown on the Figure 1.

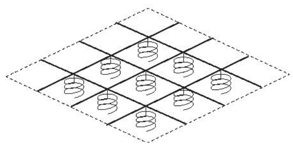

Figure 1 – A grid of strings

We assume our system consists of a finite number of strings, attached to each other in the form of a grid having the square cells. Each "internal" node is supposed to be attached to the springs and (a point of the adjacency of four strings) could move in the direction orthogonal to the coordinate plane O_{xy} (this plane gives a geometric representation of the plane, containing an initial configuration at rest). The grid experiences a resistance of these springs during movement. The nodes of the grid lying on the boundary (the boundary is marked by the dotted line on the Figure 1) are supposed to be fixed.

We will show that that under some assumptions about mass distribution, a tension of each string and elasticity coefficients of the springs the initial part of the spectrum of frequency of natural oscillations of the grid is close to analogues part of frequencies of some membrane, if the mesh (the size of square cells enclosed by strings) is small enough.

Moreover the corresponding modes of oscillations of the grid and the membrane are also close to each other.

Mathematical model. We assume that a mass distribution along each string is given by the function

$$\rho = \frac{h}{2}, \qquad (1)$$

where h is the length of the string; we assume the grid stretched over the unit square (see Figure 1) and the length h equals $1/n$ (n is number of strings in a horizontal or vertical chain of string between the dotted lines).

The grid is stretched so that the tension of each string equals

$$T_h = h. \tag{2}$$

The elasticity coefficient of each string equals to

$$k_h = h^2. \tag{3}$$

To reformulate in mathematical terms a problem of natural oscillations, we interpret a grid as a graph G_h with vertical and horizontal edges e (former strings) adjacent to each other in vertices $v = (ih, jh),\ (0 \le i, j \le n)$ (former nodes). Each edge may be parameterize by $t \in [0; h]$, i.e. we use a natural parameter – the arc length. We assume the horizontal edges are oriented from the left to the right, while the vertical edges oriented from the bottom to the top. Besides, we assume that the parameterization of each is agreed with the orientation. Under these conventions a problem of natural oscillations of the grid could easily be converted into the following boundary value problems (an analog of Sturm – Liouville's problem, see [3],[4]):

$$u_e'' + \frac{\lambda}{2} u_e = 0,\ e \in E \tag{4}$$

$$\sum_{e \succ v} u_e'(v) - hu(v) = 0,\quad v \in V_0 \tag{5}$$

$$u_e(v) = u(v),\ \ for\ e \succ v,\ v \in V_0 \tag{6}$$

$$u(v) = 0, \qquad v \in \partial G_h. \tag{7}$$

Here u_e stands for a restriction of the function $u: G_h \to R$ on the edge e. In (4) a differentiation relates to the natural parameter; here an orientation of the edge does not assume $u_e'(v)$ means the derivative in the internal direction of the edge e (from the vertex v into the interior of the edge e). A notation V_0 stands for the set of internal vertices.

Under the assumptions taken above, our mechanical system is similar (an exact meaning of the word "similar" will be discussed later in this section) to a membrane, which natural oscillations are modeled by the following boundary value problem:

$$\begin{cases} \Delta u - u + \lambda u = 0 & (8) \\ u\big|_{\partial Q} = 0, & (9) \end{cases}$$

where $Q = [0,\ 1] \times [0, 1]$.

In fact, one can easily find, that the total mass of the grid of string approximately equals to 1 for h small enough (see our assumption (1)), and the mass of the membrane, described by (8),(9), equals to 1 too. One can also see that the assumption (2) about the tensions of strings makes the grid similar to the membrane in the sense of similarity between elasticities of this two mechanical systems. In fact, the region, "covered" by a grid may be tasseled into squares with side lengths h, centered at the vertices of the graph. Each side of each square is intersected by the unique edge of the graph. If we distribute the tension of corresponding string (which equals h by magnitude and orthogonal to the side of square) along the side, we will obtain a set of squares stretched as a the cells of membrane (8),(9). Similar arguments show that the last mechanical assumption (3) about the elastic resistance of the springs, applied to the grid at its vertices, is equivalent to the exterior elastic resistance applied to the membrane (the term $-u$ in the equation (8)).

Finite-difference method. We reduce the problem (4)-(7) to the system of linear equations using finite difference method (see[5], [6], [7]). Let us give k vertical and k horizontal lines and $k+1$ edge on each line. The problem (4)-(7) looks like as following:

$$u''_{i,j}(x)+\frac{\lambda}{2}u_{i,j}(x)=0,\quad w''_{j,i}(y)+\frac{\lambda}{2}w_{j,i}(y)=0,\quad x,y\in G_h,\ i=\overline{1,k+1},\ j=\overline{1,k}, \tag{4*}$$

$$u'_{i,j}(v)+u'_{i+1,j}(v)+w'_{i,j}(v)+w'_{i,j+1}(v)-hu(v)=0,\quad v\in V_0,\ i=\overline{1,k},\ j=\overline{1,k}, \tag{5*}$$

$$u_{i,j}(v)=u_{i+1,j}(v)=w_{i,j}(v)=w_{i,j+1}(v),\ v\in V_0,\ i=\overline{1,k},\ j=\overline{1,k}, \tag{6*}$$

$$u_{1,i}(v)=0,\ u_{k+1,i}(v)=0,\quad w_{i,1}(v)=0,\quad w_{i,k+1}(v)=0,\quad v\in\partial G_h,\quad i=\overline{1,k} \tag{7*}$$

where $u_{i,j}(x)$ – the functions on horizontal edges, $w_{i,j}(y)$ – the functions on vertical edges.

First we replace the square grid of graph G_h to a finite set of points of this grid. For this purpose (see the Figure 2) we split each edge into *n* pieces by *n+1* equidistant points.

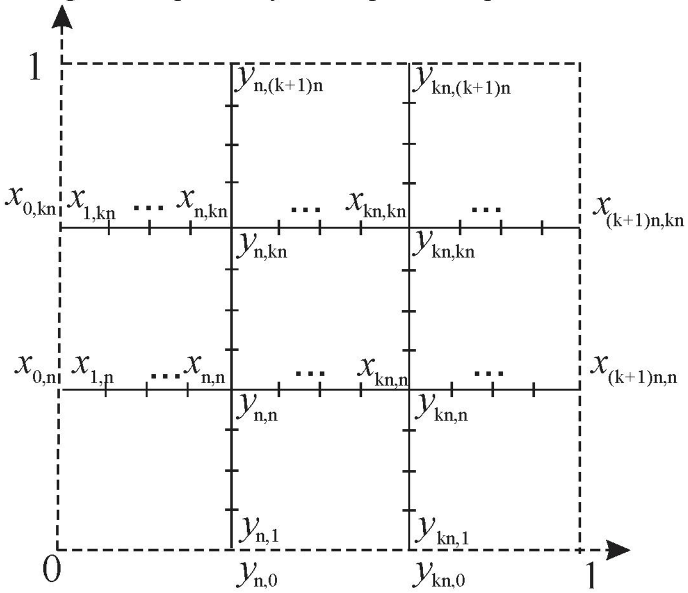

Figure 2 – Difference scheme of grid in points

Then each chain of horizontal (vertical) edges lying on and the same horizontal (vertical) line has been divided into $n\times(k+1)$ (where *k* is the number of horizontal (vertical) line). Let us denote the new grid by $\widetilde{G}_h$ which correspondent to the following conditions:

a) on horizontal lines

$$x_{0,n}=(0,n\tau),\ x_{1,n}=(\tau,n\tau),\ ...,\ x_{(k+1)n,n}=((k+1)n\tau,\ n\tau),$$

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$$x_{0,kn}=(0,kn\tau),\ x_{1,n}=(\tau,kn\tau),\ ...,\ x_{(k+1)n,n}=((k+1)n\tau,\ kn\tau),$$

b) on vertical lines

$$y_{n,0}=(n\tau,\ 0),\ y_{n,1}=(n\tau,\ \tau),\ ...,\ y_{n,(k+1)n}=(n\tau,(k+1)n\tau\ ),$$

……………………………………………………………………………

$$y_{kn,0}=(kn\tau,\ 0),\ y_{kn,1}=(kn\tau,\ \tau),\ ...,\ y_{kn,(k+1)n}=(kn\tau,\ (k+1)n\tau,)$$

where τ – is step of the new grid and the value of functions $u_{i.j}(x), w_{i.j}(y)$ at the point $x_{i,j}, y_{i,j}, \in\widetilde{G}_h$ corresponding by $\widetilde{u}_{i,j}$, $\widetilde{w}_{i,j}$. The points $x_{i,j}, y_{i,j}$ are called nodes of grid $\widetilde{G}_h$:

$$\widetilde{G}_h=\left\{x_{i-1,jn}=((i-1)\tau,\ jn\tau),\ y_{jn,i-1}=(jn\tau,\ (i-1)\tau),\ i=\overline{1,n(k+1)+1},\ j=\overline{1,k}\right\}$$

where

$$\tau = \frac{1}{(k+1)n}.$$

The selected scheme of approximation of the partial differential can be graphically represented as following:

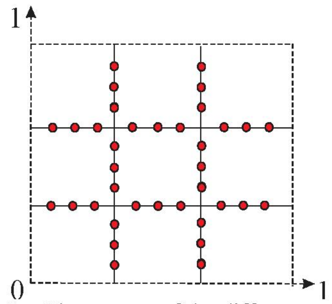


Figure 3 – The pattern of the difference scheme

In this scheme we consider only interior nodes of each edge (see Figure 3), because the condition (2*) at the interior vertices $v_{i,j} = x_{in,jn}\left(i=\overline{1,k},\ j=\overline{1,k}\right)$ break the symmetry of the matrix of the problem (4*)-(7*). We can easily determine all values at the interior vertices $v_{i,j} = x_{in,jn}$ $(i=\overline{1,k},\ j=\overline{1,k})$ of grid $\widetilde{G}_h$ by four neighbor interior nodes $x_{in-1,jn}$, $x_{in+1,jn}$, $x_{in,jn-1}$, $x_{in,jn+1}$:

$$u_{in,jn} = \frac{w_{in,jn-1} + w_{in,jn+1} + u_{in-1,jn} + u_{in-1,jn}}{4+\tau h}, \quad i=\overline{1,k}, \quad j=\overline{1,k}.$$

We replace the differential operators in (6*)-(7*) in their finite difference analogues:

$$u_{in,jn} = w_{in,jn}, \ i=\overline{1,k}, \quad j=\overline{1,k}, \qquad (6^{**})$$

$$u_{0,in} = 0, \ u_{(k+1)n,in} = 0, \quad w_{in,0} = 0, \quad w_{in,(k+1)n} = 0, \qquad i=\overline{1,k} \qquad (7^{**})$$

We approximate of derivatives of the system (4*) using the conditions (5**), (6**), (7**). First, we give analogue of the equation (4*) at the first and end nodes of each line

$$2\frac{u_{2,jn} - 2u_{1,jn}}{\tau^2} + \lambda u_{1,jn} = 0, \quad 2\frac{u_{(k+1)n-2,jn} - 2u_{(k+1)n-1,jn}}{\tau^2} + \lambda u_{(k+1)n-1,jn} = 0, \quad j=\overline{1,k},$$

$$2\frac{w_{jn,2} - 2w_{jn,1}}{\tau^2} + \lambda u_{jn,1} = 0, \quad 2\frac{w_{jn,(k+1)n-2} - 2w_{jn,(k+1)n-1}}{\tau^2} + \lambda u_{jn,(k+1)n-1} = 0, \quad j=\overline{1,k}.$$

As a result of the approximation of the partial derivatives corresponding finite difference we get the following system of linear algebraic equations at the interior nodes of each line, not including interior vertices on this lines:

$$2\frac{u_{i+1,jn} - 2u_{i,jn} + u_{i-1,jn}}{\tau^2} + \lambda u_{i,jn} = 0, \ 2\frac{w_{jn,i+1} - 2w_{jn,i} + w_{jn,i-1}}{\tau^2} + \lambda w_{jn,i} = 0,$$

$$j=\overline{1,k}, \ i=\overline{(\hat{i}-1)n+2, \hat{i}n-2}, \ \hat{i}=\overline{1,k},$$

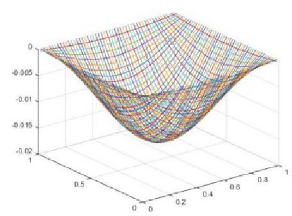
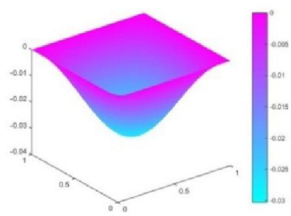
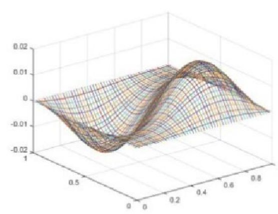
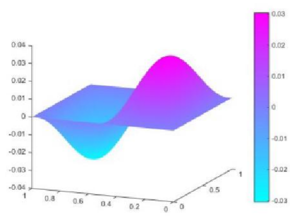
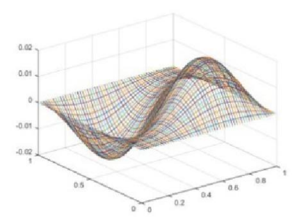
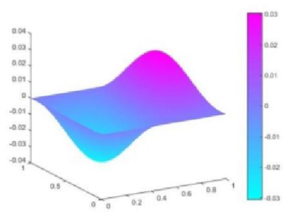
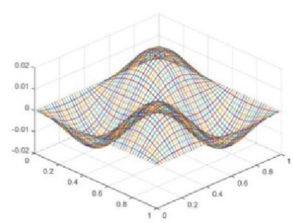
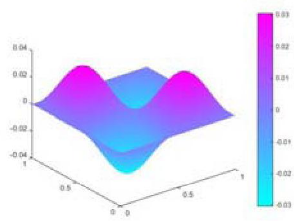
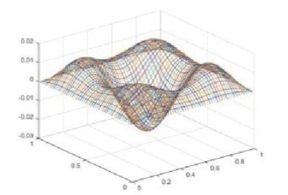
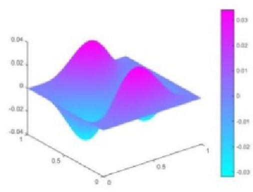
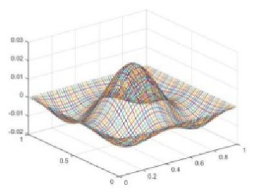
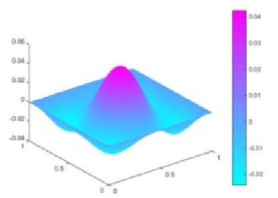
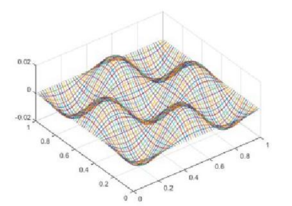
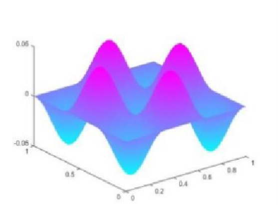
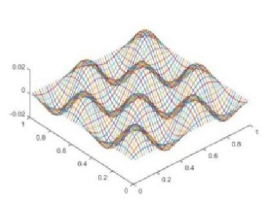
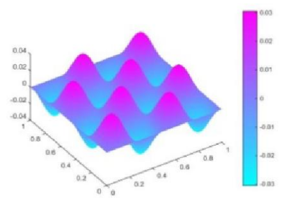
Finally, we give approximation of (1*) (see [8]) at the neighbor nodes of interior vertices $x_{in,jn}$ using the condition (5**)

$$\left[\begin{array}{l} 2\dfrac{u_{in-2,jn} - 2u_{in-1,jn}}{\tau^2} + 2\dfrac{u_{in-1,jn} + u_{in+1,jn} + w_{in,jn+1} + w_{in,jn-1}}{(4+\tau h)\tau^2} + \lambda u_{in-1,jn} = 0, \ j=\overline{1,k}, \ i=\overline{1,k}, \\ 2\dfrac{w_{in,jn-2} - 2w_{in,jn-1}}{\tau^2} + 2\dfrac{w_{in,jn-1} + w_{in,jn+1} + u_{in+1,jn} + u_{in-1,jn}}{(4+\tau h)\tau^2} + \lambda u_{in-1,jn} = 0, \ j=\overline{1,k}, \ i=\overline{1,k}. \end{array}\right.$$

Results. In this section we give some numerical results of the problem (4)-(7) and (8)-(9) by the programming language Matlab (see [9], [10], [11]). Matlab has a large number of packages, which increase efficiency of the system more times. One of such package is a PDE TOOLBOX, which intended for solving differential equations in partial derivatives and their systems (see [12], [13]). We used package PDE TOOLBOX in the calculation of the eigenvalues and eigenfunctions of the membrane. The calculation of these problems is made on a square grid $[0,1]\times[0,1]$. It was investigated eigenvalues and eigenfunctions of special grid of string (see [14], [15], [16]), which the number of vertical (horizontal) lines equal to $k = 8$. The number of edge on each line equals to 9 and the length of each edge is $h=\frac{1}{9}$, n=50, thus step of the grid $\widetilde{G}_h$ equals to $\tau = 0.002$. Let us denote step of membrane by $\widetilde{h}$. The step of membrane approximately equals to $\widetilde{h} \approx 0.0257$.

In the Table 1 the eigenfunctions of a special of grid string and the eigenfunctions of some membrane is shown. One can see that the eigenfunctions of special grid of string and membrane are similar (see ([17], [18], [19]).

Table 1 – The eigenfunctions of special grid of string and some membrane. $\hat{\lambda}_i$ – eigenvalue of string, λ_i –eigenvalue of membrane

Eigenfunction of string	**Eigenfunction of membrane**	**Eigenfunction of string**	**Eigenfunction of membrane**
 $\hat{\lambda}_1 = 21.2$	 $\lambda_1 = 20.7$	 $\hat{\lambda}_2 = 50.9$	 $\lambda_2 = 50.5$
 $\hat{\lambda}_3 = 50.9$	 $\lambda_3 = 50.5$	 $\hat{\lambda}_4 = 81.5$	 $\lambda_4 = 80$
 $\hat{\lambda}_5 = 98.3$	 $\lambda_5 = 100$	 $\hat{\lambda}_6 = 98.3$	 $\lambda_6 = 100$
 $\hat{\lambda}_{11} = 182.2$	 $\lambda_{11} = 180$	 $\hat{\lambda}_{20} = 301.8$	 $\lambda_{20} = 321$

In the Table 2 the results of eigenvalues of special grid of string and membrane, where $h = 0.1$, $\tau = 0.002$, $\widetilde{h} = 0.0257$ is shown. The eigenvalue of special grid of string closes to eigenvalue of membrane.

№	Eigenvalue of membrane λ_i	Eigenvalue of string $\hat{\lambda}_i$
1	20.7559	21.1613052346441
2	50.4513	50.9031594257733
3	50.4529	50.9031594260744
4	80.2238	81.5835446935957
5	100.1136	98.3161096946738
6	100.1148	98.3161096957769
7	130.0003	130.582745409383
8	130.0081	130.582745409516
9	169.9894	159.902826747078
10	169.9936	159.902826748113
11	180.0268	182.280004180308
12	200.0311	194.412522103279
13	200.0921	194.412522106297
14	250.3364	230.279322042279
15	250.3607	230.279322045306
16	260.3620	249.963262489939
17	260.3923	249.963262491362
18	290.6995	267.640881097907
19	290.7233	267.640881099888
20	321.1546	301.858857023673
21	341.3851	301.858857025118
22	341.5113	323.238452409586
23	371.8041	328.159780158053
24	371.8246	328.159780158240
25	402.3908	342.491053529061
26	402.4323	342.491053530374
27	412.7390	364.930055817108
28	412.8251	364.930055819259
29	453.6041	408.809859855501
30	453.6347	408.809859855804
31	504.7279	408.809859856058
32	504.9670	408.809859856083
33	505.1312	408.809859856117
34	525.6132	408.809859856134
35	525.7821	408.809859856204
36	535.7358	408.809859856494

37	535.8112	455.184680914906
38	587.5983	455.184680916734
39	587.8342	481.005232747412
40	618.6800	481.005232749098
41	618.7805	498.329459142439
42	660.1324	498.329459143523
43	660.1881	504.437824568321
44	660.3037	531.980972499042
45	660.3657	531.980972500144
46	691.5258	579.831176750581
47	691.8229	579.831176752192
48	733.7459	606.618121773322
49	743.8976	606.618121774464
50	743.9673	638.344087858107
51	754.2927	638.344087860762
52	755.1989	702.180188253403

Table 2 – Eigenvalues of a special grid of string and some membrane. $\hat{\lambda}_i$ – eigenvalue of string, λ_i –eigenvalue of membrane

Conclusion. The numerical results about eigenvalues and corresponding eigenfunctions of a special grid of strings in the square $[0,\ 1]\times[0,1]$ have been researched. It was shown that the eigenvalues of a special grid of strings partially close to the eigenvalues of membrane (see [20]) and the eigenfunctions are similar.

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ЧИСЛЕННЫЕ РЕЗУЛЬТАТЫ СПЕЦИАЛЬНОЙ СЕТКИ ИЗ СТРУН

Аннотация. В данной работе рассмотриваются некоторые численные результаты о собственных колебаний специальной сетки из струн. Задача о струне частот собственных колебаний такой сетки сводится к задаче Штурма-Лиувилля на сетки. Для изучения колебаний специальной сетки из струн используется метод конечных разностей. Основные результаты работы даются на языке программирования Matlab.

Ключевые слова: струна, вершина, ребра, узел, собственное число, собственный вектор

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АРНАЙЫ ІШЕКТЕН ЖАСАЛҒАН ТОРДЫҢ САНДЫҚ НӘТИЖЕЛЕРІ

Аннотация. Бұл жұмыста ішектен жасалған арнайы тордың меншікті тербелісінің сандық нәтижелері қарастырылады. Мұндай ішектен жасалған арнайы тордың меншікті тербеліс жиілігі туралы есеп тордағы Штурм-Лиувилл есебіне келтіріледі. Ішектен жасалған арнайы тор тербелісін зерттеу барысында ақырлы-айырымдық әдіс қолданылған. Жұмыстың негізгі алынған нәтижелері Matlab программалау тілінде зерттелген.

Түйін сөздер: ішек, төбе, қабырға, түйін, меншікті мән, меншікті вектор.