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ANALYTICAL SOLUTION OF HEAT EQUATION WITH MOVING BOUNDARY NOT TANGENT TO AXIS BY HEAT POLYNOMIALS

Abstract. It was found the analytical solution of heat equation with moving boundary with a method of heat polynomials for finding the coefficients.
Key words: Integral Error Functions.

Introduction

It's well known that a wide range of transient phenomena in fields of low-temperature plasma, filtration and other evolutionary processes which are associated with phase transformations lead to the necessity of solving heat and mass transfer problems with free moving inter-phase boundaries. Development of analytical methods of solution of free boundary problems are very important for analysis of dynamics of mentioned phenomena specifically phenomena occurring in electrical contacts.

The well-known analytical method is based on the representation of a solution in the form of heat potential with following reduction of the given problem to integral equation [1]. However if the domain with moving boundary degenerates into a point at the initial time, the integral equations become singular and cannot be solved by Picard's iteration method. Asymptotic properties of such equations have been investigated in [2]. Auto-model case when the boundary $\alpha(t)$ is moving according to the law $\alpha(t)=c\sqrt{t}$ is considered in [3] where analytical solution is found. Solution of the problem with $\alpha(t)=ct$ is represented by Heat potentials method in [4]. In this study we use heat polynomials which are elaborated from Integral Error Functions and its properties to solve given boundary value problem.

Problem statement

Definition: The class functions M_β is defined by formula: $f(t)\in M_\beta(h)$ if $f(t)$ is continuous on the interval $(0,t)$ and $\lim\limits_{t\to 0}\frac{f(t)}{t^\beta}=h=const$, where β is any real number.

The main problem can be formulated as following. It is required to find the solution of the heat equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$

in the domain $D:(t>0, 0<x<\alpha(t))$, where $\alpha(\mathrm{t})=\mathrm{ct}$, degenerates at the initial time: $\alpha(0)=0$, $\varphi(0)=0$ and satisfies the initial condition

$$u(0,0)=0 \tag{2}$$

and the boundary conditions

$$u(0,t)=\varphi(t) \tag{3}$$

$$u(\alpha(t),t)=U-const \tag{4}$$

Problem solution:

Solution of equation (1) can be represented in the following form

$$u(x,t)=\sum_{n=0}^{\infty}\left\{A_n\cdot t^n\left[i^{2n}erfc\frac{x}{2a\sqrt{t}}+i^{2n}erfc(\frac{-x}{2a\sqrt{t}})\right]+B_n\cdot t^{\frac{2n+1}{2}}\left[i^{2n+1}erfc\frac{x}{2a\sqrt{t}}-i^{2n+1}erfc(\frac{-x}{2a\sqrt{t}})\right]\right\} \quad (5)$$

or in the form of heat polynomials

$$u(x,t)=\sum_{n=0}^{\infty}\left\{A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\right\} \quad (6)$$

$$\begin{aligned}
u(\mathrm{x},\mathrm{t}) &= \mathrm{A}_0\,\beta_{0,0}+\\
&A_2(\mathrm{x}^2\,\beta_{2,0}+\mathrm{t}\,\beta_{2,1})+\\
&\mathrm{A}_4(\mathrm{x}^4\,\beta_{4,0}+\mathrm{x}^2\mathrm{t}\,\beta_{4,1}+\mathrm{t}^2\,\beta_{4,2})+\ldots\\
&+\mathrm{A}_{2n}(\mathrm{x}^{2n}\,\beta_{2n,0}+\mathrm{x}^{2n-2}\mathrm{t}\,\beta_{2n,1}+\ldots+\mathrm{x}^2\mathrm{t}^{n-1}\,\beta_{2n,n-1}+\mathrm{t}^n\,\beta_{2n,\mathrm{n}})+\ldots\\
&A_1x\beta_{1,0}+\\
&A_3(\mathrm{x}^3\,\beta_{3,0}+\mathrm{xt}\,\beta_{3,1})+\\
&+\mathrm{A}_5(\mathrm{x}^5\,\beta_{5,0}+\mathrm{x}^3\mathrm{t}\,\beta_{5,1}+\mathrm{xt}^2\,\beta_{5,2})+\ldots\\
&+A_{2n+1}(\mathrm{x}^{2n+1}\,\beta_{2n+1,0}+\mathrm{x}^{2n-1}\mathrm{t}\,\beta_{2n+1,1}+\ldots+x^3\,\mathrm{t}^{n-1}\,\beta_{2n+1,\mathrm{n}-1}+\mathrm{xt}^n\,\beta_{2n+1,\mathrm{n}})+\ldots
\end{aligned} \quad (7)$$

where

$$\beta(\mathrm{n},\mathrm{m})=\frac{1}{2^{n+m-1}m!(\mathrm{n}-2\,\mathrm{m})!}$$

It's easy to see that if we use expression (6) for x=0, expand function φ(t) into Maclaurin's series and combine like terms on the left side in (3), then we have

$$\sum_{n=0}^{\infty}A_{2n}\beta_{2n,\mathrm{n}}t^n=\sum_{n=0}^{\infty}\frac{\varphi^{(\mathrm{n})}(0)}{n!}t^n,$$

which implies following formula for A_{2n}

$$A_{2n}\beta_{2n,\mathrm{n}}=\varphi_n \quad (8)$$

where $\varphi_n=\dfrac{\varphi^{(\mathrm{n})}(0)}{n!}$

Utilizing (7) for $x=ct$ from (5) we have

$$\begin{aligned}
u(\mathrm{ct},\mathrm{t}) &= \mathrm{A}_0\,\beta_{0,0}+\\
&A_2(\mathrm{c}^2\,t^2\beta_{2,0}+\mathrm{t}\,\beta_{2,1})+\\
&\mathrm{A}_4(\mathrm{c}^4\,t^4\beta_{4,0}+c^2t^3\beta_{4,1}+\mathrm{t}^2\,\beta_{4,2})+\ldots\\
&+\mathrm{A}_{2n}(\mathrm{c}^{2n}\,t^{2n}\beta_{2n,0}+c^{2n-2}\,\mathrm{t}^{2n-1}\,\beta_{2n,1}+\ldots+c^2\,\mathrm{t}^{n+1}\,\beta_{2n,n-1}+\mathrm{t}^n\,\beta_{2n,\mathrm{n}})+\ldots\\
&A_1ct\beta_{1,0}+\\
&A_3(\mathrm{c}^3\,t^3\beta_{3,0}+c\,\mathrm{t}^2\,\beta_{3,1})+\\
&+\mathrm{A}_5(\mathrm{c}^5\,t^5\beta_{5,0}+c^3\,\mathrm{t}^4\,\beta_{5,1}+c\,\mathrm{t}^3\,\beta_{5,2})+\ldots\\
&+A_{2n+1}(\mathrm{c}^{2n+1}\,t^{2n+1}\beta_{2n+1,0}+c^{2n-1}\,\mathrm{t}^{2n}\,\beta_{2n+1,1}+\ldots+c^3\,\mathrm{t}^{n+2}\,\beta_{2n+1,\mathrm{n}-1}+c\,\mathrm{t}^{n+1}\,\beta_{2n+1,\mathrm{n}})+\ldots=U
\end{aligned} \quad (9)$$

If we expand ψ(t) into Maclaurin's series combine like terms of left side and then take derivatives of both sides of (8) it is easy to derive from below relations following recurrent formula (10) for coefficient A_{2n+1}

$$A_0\beta_{0,0} = U$$

$$A_1 c\beta_{1,0} + A_2\beta_{2,1} = 0$$

$$A_2 c^2\beta_{2,0} + A_3 c\beta_{3,1} + A_4\beta_{4,2} = 0$$

$$A_3 c^3\beta_{3,0} + A_4 c^2\beta_{4,1} + A_5 c\beta_{5,2} + A_6\beta_{6,3} = 0$$

$$A_4 c^4\beta_{4,0} + A_5 c^3\beta_{5,1} + A_6 c^2\beta_{6,2} + A_7 c\beta_{7,3} + A_8\beta_{8,4} = 0$$

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$$A_n c^n\beta_{n,0} + A_{n+1}c^{n-1}\beta_{n+1,1} + A_{n+2}c^{n-2}\beta_{n+2,2} + \ldots + A_{2n-1}c\beta_{2n-1,n-1} + A_{2n}\beta_{2n,n} = 0$$

$$A_{n+1}c^{n+1}\beta_{n+1,0} + A_{n+2}c^n\beta_{n+2,1} + A_{n+3}c^{n-1}\beta_{n+3,2} + \ldots + A_{2n+1}c\beta_{2n+1,n} + A_{2n+2}\beta_{2n+2,n+1} = 0$$

$$A_{2n+1}c\beta_{2n+1,n} = -A_{2n+2}\beta_{2n+2,n+1} - \sum_{m=n+1}^{2n} A_m c^{2n+2-m}\beta_{m,m-n-1} \tag{10}$$

Convergence

Let's take $t = t_0$ to prove convergence of series (3) in the interval $0 < x < \alpha(t)$. From (8) and (10) we get following even and odd coefficients respectively

$$A_{2n} = \frac{\varphi_n}{\beta_{2n,n}} = \frac{\varphi^{(n)}(0)}{n!}\cdot 2^{3n-1}\cdot n!(2n-2n)! = \varphi^{(n)}\cdot 2^{3n-1}$$

$$A_{2n+1} = \frac{1}{c\beta_{2n+1,n}}\left[-A_{2n+2}\varphi^{(n)}\cdot 2^{3n-1}\beta_{2n+2,n+1} - \sum_{m=n+1}^{2n} A_n c^{2n+2-m}\cdot\beta_{m,m-n-1}\right]$$

where

$$\beta_{2n+2,n+1} = \frac{1}{2^{3n+2}(n+1)!(2n+2-2(n+1))!},$$

$$\beta_{m,m-n-1} = \frac{1}{2^{2m-n-2}(m-n-1)!(2n+2-m)!}$$

To prove convergence of (3) we have to demand convergence of

$$\sum_{n=0}^{\infty}\left\{A_{2n+1}\sum_{m=0}^{n}(ct_0)^{2n-2m+1}t_0^m\beta_{2n+1,m}\right\} \text{ and } \sum_{m=0}^{n}(ct_0)^{2n-2m+1}t_0^m\beta_{2n+1,m} \text{ respectively}$$

By d'Alambert's convergence test from last sentence of series (9) we get

$$\frac{c^{2n+1}\beta_{2n+1,0}}{c^{2n-1}\beta_{2n+1,1}} = c^2\frac{2^{2n}\cdot(2n+1)!}{2^{2n+1}(2n-1)!} = c^2\frac{1}{2}\cdot(2n)(2n+1) = c^2\cdot n(2n+1) < 1$$

$$\Rightarrow c < \frac{1}{\sqrt{n(2n+1)}}$$

For $t = t_0$ series (3) is bounded and let

$$A_{2n+1}t_0^{n+1}\sum_{m=0}^{n}t_0^m\beta_{2n+1,n-m}c^{2m+1} < C_1$$

$$A_{2n+1} < \frac{C_1}{t_0^{n+1} \sum_{m=0}^{n} t_0^m \beta_{2n+1,n-m} c^{2m+1}}$$

For any value $t < t_0$ let's multiply both sides of latter expression by

$$t^{n+1} \sum_{m=0}^{n} t^m \beta_{2n+1,n-m} c^{2m+1}$$

$$A_{2n+1} t^{n+1} \sum_{m=0}^{n} t^m \beta_{2n+1,n-m} c^{2m+1} < C_1 \frac{t^{n+1} \sum_{m=0}^{n} t^m \beta_{2n+1,n-m} c^{2m+1}}{t_0^{n+1} \sum_{m=0}^{n} t_0^m \beta_{2n+1,n-m} c^{2m+1}} < C_1 \left(\frac{t}{t_0} \right)^{n+1}$$

Take the sum of the both sides

$$\Rightarrow \sum_{n=0}^{\infty} A_{2n+1} \sum_{m=0}^{n} t^{2n+1} \beta_{2n+1,n-m} c^{2m+1} < C_1 \sum_{n=0}^{\infty} \left(\frac{t}{t_0} \right)^{n+1}$$

Thus convergence is proved

Conclusion

Problem (1)-(4) is solved by heat polynomials. Coefficients A_{2n}, A_{2n+1} can be calculated from recurrent formulas (8) and (10) respectively. Convergence of solution series is proved.

In the second problem, we introduced the heat equation with a moving boundary, which degenerates at the initial moment of time represented in explicit analytic form. The developed method is based on the integral functions of the error and its properties. The main idea was to find the coefficients of the linear combination of the integral error functions that a priori satisfy the heat equation.

REFERENCES

[1] Tihonov A.N. , Samarskij A.A. Uravnenija matematicheskoj fiziki. Gostehteorizdat. Moskva, 1951

[2] Harin S.N. // Teplovye processy v jelektricheskih kontaktah i svjazannye s nimi singuljarnye integral'nye uravnenija. Avtoreferat dissertacii. Alma-Ata, 1968

[3] Harin S.N. // O teplovyh zadachah s podvizhnoj granicej. Izvestija AN Kaz SSR, ser. fiz.- mat. nauk, № 3, 1965

[4] S.N. Kharin, M.M. Sarsengeldin. Analytical solution of heat equation with moving boundary not tangent to coordinate axis. Herald of NAS RK, 2011, 3-rd ed. Phys-math, pp. 3-8

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АНАЛИТИЧЕСКОЕ РЕШЕНИЕ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ С ДВИЖУЩИМИСЯ ГРАНИЦАМИ НЕ КАСАЮЩИЕСЯ ОСИ ТЕПЛОВЫМИ ПОЛИНОМАМИ

Аннотация: Найдено аналитическое решение уравнения теплопроводности с движущей границей методом тепловых полиномов для нахождения коэффициентов.

Ключевые слова: Интегральная Функция Ошибок.

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ҚОЗҒАЛМАЛЫ ШЕКАРАСЫ БАР ОСЬКЕ ТИМЕЙТІН ЖЫЛУӨТКІЗГІШТІК ТЕНДЕУІНІҢ ЖЫЛУ КӨПМҮШЕЛЕРІ АРҚЫЛЫ АНАЛИТИКАЛЫҚ ШЕШІМІ

Аннотация: Қозғалмалы шекарасы бар жылуөткізгіштік тендеуінің жылу көпмүшелері арқылы аналитикалық шешімі және жылуөткізгіштік тендеуін қанағаттандыратын коэффициенттері табылды.

Тірек сөздер: Интегралды Қателіктер Функциясы.