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APPLICATION OF POLYGONAL METHOD TO SOLVE OF PERIODIC PROBLEM FOR LOADED AND INTEGRO-DIFFERENTIAL PARABOLIC EQUATIONS

Abstract. In the first section, we investigate the periodic boundary value problem for a loaded parabolic equations in a rectangular domain. Using the polygonal method we construct of an algorithms for finding solutions of the periodic boundary value problem for loaded parabolic equations. And the convergence of algorithms is proved. Conditions of unique solvability of the investigated problem are establiched in the terms of initial data. In the second section, we investigate the periodic boundary value problem for parabolic integro-differential equation in a rectangular domain. The polygonal method develops on parabolic integro-differential equation. Algorithms for finding solutions of the periodic boundary value problem for parabolic integro-differential equations are constructed, and their convergence is proved. Conditions of unique solvability of the investigated problem are establiched in the terms of initial data.

Key words: periodic problem, loaded parabolic equations, integro-differential parabolic equations, polygonal method, algorithm, unique solvability.

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Loaded partial differential equations parabolic type arise in the study of various processes of physics, chemistry, biology, ecology and others [1-5]. Another important class of problems closely related to evolutionary integro-differential equations in partial derivatives are the parabolic integro-differential equations and boundary value problems for them [6-12]. The conditions for unique solvability, the assessment of solutions and their derivatives in terms of the geometric characteristics of the coefficients, the right-hand side, boundary values, and the region where linear boundary value problems for loaded partial differential equations are given, find in numerous applications in the qualitative theory of boundary value problems.

In [13] by using the polygonal method on the spatial variable the boundary value problem for a parabolic equation has been reduced to the solving of family of the Cauchy problem for a system of ordinary differential equations. Using the parametrization method [14] there were established the effective estimates of solutions through the initial data [15]. This approach will be developed on the parabolic loaded and integro-differential equations. There will be developed a constructive method for the solving of periodic boundary value problems for parabolic loaded and integro-differential equations.

By polygonal method on spatial variable the periodic and nonlocal boundary value problems for parabolic integro-differential equations will be reduced to a family of Cauchy problems for systems of ordinary integro-differential equations. On the basis of parameterization method the algorithms for finding the solution will be built and the conditions of unique solvability of the considered problem will be established.

1. Periodic boundary value problem for loaded parabolic equations

We consider a periodic boundary value problem for a loaded parabolic equation

$$\frac{\partial u}{\partial t}=a(t,x)\frac{\partial^2 u}{\partial x^2}+c(t,x)u(t,x)+\alpha(t)u(\theta,x)+f(t,x),\ (t,x)\in\Omega=(0,T)\times(0,\omega), \quad (1.1)$$

$$u(0,x)=u(T,x),\qquad x\in[0,\omega], \quad (1.2)$$

$$u(t,0)=\psi_0(t),\qquad u(t,\omega)=\psi_1(t),\qquad t\in[0,T], \quad (1.3)$$

where $a(t,x)\ge a_0>0$, $c(t,x)\le 0$ $c(t,x)\le 0$, $f(t,x)$ are continuous with respect to t, Holder continuous with respect to x, $\alpha(t)$ is continuous function on $[0,T]$. It is assumed that the functions $\psi_0(t)$, $\psi_1(t)$ are sufficiently smooth and satisfy the matching conditions $\psi_0(0)=\psi_0(T)$, $\psi_1(0)=\psi_1(T)$.

The parametrization method is applied to the periodic boundary value problem (1.1)-(1.3). Let $\lambda(x)=u(0,x)$ and in a problem (1.1)-(1.3) we will carry out replacement $u(t,x)=\lambda(x)+\tilde{u}(t,x)$, where $\tilde{u}(t,x)$ is a new unknown function. Then the periodic boundary value problem (1.1)-(1.3) reduce to the following problem

$$\frac{\partial\tilde{u}}{\partial t}=a(t,x)\frac{\partial^2\tilde{u}}{\partial x^2}+c(t,x)\tilde{u}(t,x)+\alpha(t)\tilde{u}(\theta,x)+$$

$$+a(t,x)\ddot{\lambda}(x)+c(t,x)\lambda(x)+\alpha(t)\lambda(x)+f(t,x),\ (t,x)\in\Omega, \quad (1.4)$$

$$\tilde{u}(0,x)=0,\qquad x\in[0,\omega], \quad (1.5)$$

$$\tilde{u}(t,0)+\lambda(0)=\psi_0(t),\qquad \tilde{u}(t,\omega)+\lambda(\omega)=\psi_1(t),\qquad t\in[0,T], \quad (1.6)$$

$$\tilde{u}(T,x)=0,\qquad x\in[0,\omega]. \quad (1.7)$$

From conditions (1.5) and (1.6) follows

$$\lambda(0)=\psi_0(0),\ \lambda(\omega)=\psi_1(0).$$

The problem (1.4)-(1.7) is an initial-boundary value problem for a loaded parabolic equation with a parameter. An algorithm for finding the solution of the problem (1.4)-(1.7) is constructed, which consists of two stages: 1) the solving of the initial-boundary value problem for the loaded parabolic equation (1.4)-(1.6) at the fixed parameter by means of a justification polygonal method [42-43] ; 2) determination of the parameter from the relation (1.7).

The first stage of the algorithm. We consider an auxiliary initial-boundary value problem for the loaded parabolic equation

$$\frac{\partial\tilde{u}}{\partial t}=a(t,x)\frac{\partial^2\tilde{u}}{\partial x^2}+c(t,x)\tilde{u}(t,x)+\alpha(t)\tilde{u}(\theta,x)+\tilde{f}(t,x),\ (t,x)\in\Omega, \quad (1.8)$$

$$\tilde{u}(0,x)=0,\qquad x\in[0,\omega], \quad (1.9)$$

$$\tilde{u}(t,0)=\psi_0(t)-\psi_0(0),\qquad \tilde{u}(t,\omega)=\psi_1(t)-\psi_1(0),\ t\in[0,T], \quad (1.10)$$

where function $\tilde{f}(t,x)$ is continuous with respect to t, and is Holder continuous with respect to x.

The scheme of the polygonal method with respect to the problem (1.8)-(1.10). We take $h>0$ and make a discretization by x: $x_i=ih$, $i=\overline{0,N}$, $Nh=\omega$, $\tilde{u}_i(t)=\tilde{u}(t,ih)$. The problem (1.8)-(1.10) is replaced by the following form

$$\frac{\partial\tilde{u}_i}{\partial t}=a_i(t)\frac{\tilde{u}_{i+1}-2\tilde{u}_i+\tilde{u}_{i-1}}{h^2}+c_i(t)\tilde{u}_i+\alpha(t)\tilde{u}_i(\theta)+\tilde{f}_i(t),\ \tilde{u}_i(0)=0,\ i=\overline{0,N}, \quad (1.11)$$

$$\tilde{u}_0(t)=\psi_0(t)-\psi_0(0),\qquad \tilde{u}_N(t)=\psi_1(t)-\psi_1(0),\ t\in[0,T]. \quad (1.12)$$

Owing to linearity of system for $\forall h>0$, there exists a unique solution of problem (1.11): $\{\tilde{u}_1(t),...,\tilde{u}_{N-1}(t)\}$ defined on $[0,T]$.

Taking the functions $\widetilde{u}_{i+1}$, $\widetilde{u}_{i-1}$ and the loaded term to the right-hand side of every i th equation of the system (1.11), we applied the estimate from [15]:

$$\|\widetilde{u}_i\| = \max_{t\in[0,T]}\{|\widetilde{u}_i(t)|\} \le \frac{1}{2}\|\widetilde{u}_{i-1}(t)\| + \frac{1}{2}\|\widetilde{u}_{i+1}(t)\| + \frac{1}{2}\left\|\frac{\alpha(t)}{a_i(t)}\widetilde{u}_i(\theta)\right\|h^2 + \frac{1}{2}\left\|\frac{\widetilde{f}_i(t)}{a_i(t)}\right\|h^2.$$

Let $\xi_i = \|\widetilde{u}_i\|$. Then, we obtain the following estimate

$$\xi_i \le \frac{1}{2}\xi_{i-1} + \frac{1}{2}\xi_{i+1} + \frac{1}{2}\left\|\frac{\alpha(t)}{a_i(t)}\right\|h^2\xi_i + \frac{1}{2}\left\|\frac{\widetilde{f}_i(t)}{a_i(t)}\right\|h^2,\ i=\overline{1,N-1}. \tag{1.13}$$

Suppose that $\frac{1}{2}\max_{t\in[0,T]}\left\|\frac{\alpha(t)}{a_i(t)}\right\|h^2 \le \chi < 1$, $i=\overline{1,N-1}$, from inequality (1.13), we have

$$\xi_i \le \frac{1}{2(1-\chi)}\xi_{i-1} + \frac{1}{2(1-\chi)}\xi_{i+1} + \frac{1}{2(1-\chi)}\left\|\frac{\widetilde{f}_i(t)}{a_i(t)}\right\|h^2,\ i=\overline{1,N-1}. \tag{1.14}$$

Next, using sweep up and down, from (1.14) we get

$$\|\widetilde{u}_i\| \le \frac{N-i}{N(1-\chi)}\|\widetilde{\psi}_0\| + \frac{i}{N(1-\chi)}\|\widetilde{\psi}_1\| + \frac{N-i}{N(1-\chi)}\sum_{j=1}^{i}\left\|j\frac{\widetilde{f}_j(t)}{a_j(t)}\right\|h^2 + $$
$$+\frac{i}{N(1-\chi)}\sum_{j=i+1}^{N-1}\left\|(N-j)\frac{\widetilde{f}_j(t)}{a_j(t)}\right\|h^2 \le K_1,$$

where $\widetilde{\psi}_0(t)=\psi_0(t)-\psi_0(0)$, $\widetilde{\psi}_1(t)=\psi_1(t)-\psi_1(0)$.

From this inequality it follows the next assertion

Theorem 1.1. *Let*
a) the assumptions with respect to the data of problem (1.1)-(1.3) are fulfilled;
b) the inequality $\frac{1}{2}\max_{t\in[0,T]}\left\|\frac{\alpha(t)}{a_i(t)}\right\|h^2 \le \chi < 1$ *is valid, where* $a_i(t)=a(t,ih)$, $i=\overline{0,N}$.
Then problem (1.8)—(1.10) has a unique classical solution $\widetilde{u}^*(t,x)$*, and for it the estimate holds:*

$$\max_{t\in[0,T]}|\widetilde{u}^*(t,x)| \le \frac{\omega-x}{\omega(1-\chi)}\max_{t\in[0,T]}|\psi_0(t)-\psi_0(0)| + \frac{x}{\omega(1-\chi)}\max_{t\in[0,T]}|\psi_1(t)-\psi_1(0)| +$$
$$+\frac{\omega-x}{\omega(1-\chi)}\int_0^x z\cdot\max_{t\in[0,T]}\left|\frac{\widetilde{f}(t,z)}{a(t,z)}\right|dz + \frac{x}{\omega(1-\chi)}\int_x^\omega(\omega-z)\cdot\max_{t\in[0,T]}\left|\frac{\widetilde{f}(t,z)}{a(t,z)}\right|dz.$$

Integrating equation (1.8) by variable t and accounting condition (1.10), we have

$$\widetilde{u}(t,x) = \int_0^t a(\tau,x)\frac{\partial^2\widetilde{u}(\tau,x)}{\partial x^2}d\tau + \int_0^t a(\tau,x)d\tau\cdot\ddot{\lambda}(x) + \int_0^t c(\tau,x)\widetilde{u}(\tau,x)d\tau +$$
$$+\int_0^t c(\tau,x)d\tau\cdot\lambda(x) + \int_0^t \alpha(\tau)d\tau\cdot\widetilde{u}(\theta,x) + \int_0^t\alpha(\tau)d\tau\cdot\lambda(x) + \int_0^t f(\tau,x)d\tau. \tag{1.15}$$

From expression (1.15), we determine the value of function $\widetilde{u}(t,x)$ for $t=\theta$:

$$\tilde{u}(\theta,x)=\frac{1}{1-\int_0^\theta \alpha(\tau)d\tau}\left\{\int_0^\theta a(\tau,x)\frac{\partial^2\tilde{u}(\tau,x)}{\partial x^2}d\tau+\int_0^\theta a(\tau,x)d\tau\cdot\ddot{\lambda}(x)+\int_0^\theta c(\tau,x)\tilde{u}(\tau,x)d\tau+\right.$$

$$\left.+\int_0^\theta c(\tau,x)d\tau\cdot\lambda(x)+\int_0^\theta \alpha(\tau)d\tau\cdot\lambda(x)+\int_0^\theta f(\tau,x)d\tau\right\}.$$

Here, we suppose that $\int_0^\theta \alpha(\tau)d\tau\neq 1$ and introduce the notation $\beta(\theta)=\dfrac{1}{1-\int_0^\theta \alpha(\tau)d\tau}$. Then, the expression (1.15) has the following form

$$\tilde{u}(t,x)=\int_0^t a(\tau,x)\frac{\partial^2\tilde{u}(\tau,x)}{\partial x^2}d\tau+\int_0^t a(\tau,x)d\tau\cdot\ddot{\lambda}(x)+\int_0^t c(\tau,x)\tilde{u}(\tau,x)d\tau+$$

$$+\int_0^t c(\tau,x)d\tau\cdot\lambda(x)+\int_0^t \alpha(\tau)d\tau\cdot\lambda(x)+\int_0^t f(\tau,x)d\tau+$$

$$+\int_0^t \alpha(\tau)d\tau\cdot\beta(\theta)\left\{\int_0^\theta a(\tau,x)\frac{\partial^2\tilde{u}(\tau,x)}{\partial x^2}d\tau+\int_0^\theta c(\tau,x)\tilde{u}(\tau,x)d\tau+\right.$$

$$\left.+\int_0^\theta a(\tau,x)d\tau\cdot\ddot{\lambda}(x)+\int_0^\theta c(\tau,x)d\tau\cdot\lambda(x)+\int_0^\theta \alpha(\tau)d\tau\cdot\lambda(x)+\int_0^\theta f(\tau,x)d\tau\right\}. \quad (1.16)$$

From expression (1.16), we determine the value of function $\tilde{u}(t,x)$ for $t=T$ and replace in the condition (1.7):

$$\left[\int_0^T a(\tau,x)d\tau+\int_0^T \alpha(\tau)d\tau\beta(\theta)\int_0^\theta a(\tau,x)d\tau\right]\ddot{\lambda}(x)=$$

$$=-\left[\int_0^T c(\tau,x)d\tau+\int_0^T \alpha(\tau)d\tau+\int_0^T \alpha(\tau)d\tau\beta(\theta)\left\{\int_0^\theta c(\tau,x)d\tau+\int_0^\theta \alpha(\tau)d\tau\right\}\right]\cdot\lambda(x)-$$

$$-\int_0^T a(\tau,x)\frac{\partial^2\tilde{u}(\tau,x)}{\partial x^2}d\tau-\int_0^T c(\tau,x)\tilde{u}(\tau,x)d\tau-$$

$$-\int_0^T \alpha(\tau)d\tau\beta(\theta)\left[\int_0^\theta a(\tau,x)\frac{\partial^2\tilde{u}(\tau,x)}{\partial x^2}d\tau+\int_0^\theta c(\tau,x)\tilde{u}(\tau,x)d\tau\right]-$$

$$-\int_0^T f(\tau,x)d\tau-\int_0^T \alpha(\tau)d\tau\beta(\theta)\int_0^\theta f(\tau,x)d\tau. \quad (1.17)$$

At fixed $\tilde{u}(t,x)$ the relation (1.17) with condition $\lambda(0)=\psi_0(0)$, $\lambda(\omega)=\psi_1(0)$ is two-point boundary value problem for differential equation second order with respect to λ.

The second stage of the algorithm. We consider the auxiliary problem on the interval $[0,\omega]$

$$\left[\int_0^T a(\tau,x)d\tau+\int_0^T \alpha(\tau)d\tau\beta(\theta)\int_0^\theta a(\tau,x)d\tau\right]\ddot{\lambda}(x)=$$

$$=-\left[\int_0^T c(\tau,x)d\tau+\int_0^T \alpha(\tau)d\tau+\int_0^T \alpha(\tau)d\tau\beta(\theta)\left\{\int_0^\theta c(\tau,x)d\tau+\int_0^\theta \alpha(\tau)d\tau\right\}\right]\cdot\lambda(x)-g(x), \quad (1.18)$$

where $g(x)$ is continuous function on $[0,\omega]$.

From the matching condition we obtain

$$\lambda(0)=\psi_0(0),\ \lambda(\omega)=\psi_1(0). \quad (1.19)$$

The problem (1.18), (1.19) is a two-point boundary value problem for a second-order differential equation with respect to a function $\lambda(x)$.

Let

$$\widetilde{a}(x)=\int_0^T a(\tau,x)d\tau+\int_0^T \alpha(\tau)d\tau\beta(\theta)\int_0^\theta a(\tau,x)d\tau,$$

$$\widetilde{b}(x)=\int_0^T c(\tau,x)d\tau+\int_0^T \alpha(\tau)d\tau+\int_0^T \alpha(\tau)d\tau\beta(\theta)\left\{\int_0^\theta c(\tau,x)d\tau+\int_0^\theta \alpha(\tau)d\tau\right\},$$

The following assertion given of a condtions unique solvability to problem (1.18), (1.19).

Theorem 1.2. *Let*

a) the assumptions with respect to the data of problem (1.1)-(1.3) are fulfilled;

b) the condition $\int_0^\theta \alpha(\tau)d\tau\neq 1$ *is valid;*

c) the condition $\int_0^T a(\tau,x)d\tau+\int_0^T \alpha(\tau)d\tau\beta(\theta)\int_0^\theta a(\tau,x)d\tau\neq 0$ *is valid for all* $x\in[0,\omega]$;

d) the inequality holds: $\max\left(1,\frac{2}{\omega}\right)\left[e^{\alpha\omega}-1-\alpha\omega\right]<1$, *wher*

e $\alpha=\max\left(1,1/\max_{x\in[0,\omega]}|\widetilde{a}(x)|\right)\cdot\max\left(\max_{x\in[0,\omega]}|b(x)|,1\right)$.

Then problem (1.18), (1.19) has a unique solution $\lambda^*(x)$.

Theorems 1.1 and 1.2 given of the conditions unique solvability of auxiliary problems (1.8)-(1.10) and (1.18), (1.19) in the terms of initial data.

On each step of the algorithm: 1) the initial-boundary value problem for the parabolic integro-differential equation is solved at a fixed $\lambda(x)$; 2) a two-point boundary value problem for a second-order differential equation is solved at a fixed $\widetilde{u}(t,x)$.

Conditions of Theorems 1.1 and 1.2 guarantee of realizable and convergence of proposed algorithm.

2. Periodic boundary value problem for parabolic integro-differential equations

We consider periodic boundary value problem for parabolic integro-differential equations

$$\frac{\partial u}{\partial t}=a(t,x)\frac{\partial^2 u}{\partial x^2}+c(t,x)u(t,x)+\alpha(t)\int_0^t u(\tau,x)d\tau+f(t,x),\ (t,x)\in\Omega, \quad (2.1)$$

$$u(0,x)=u(T,x),\qquad x\in[0,\omega], \quad (2.2)$$

$$u(t,0)=\psi_0(t),\qquad u(t,\omega)=\psi_1(t),\qquad t\in[0,T], \quad (2.3)$$

where $a(t,x)\geq a_0>0$, $c(t,x)\leq 0$, $f(t,x)$ - are continuous with respect to t, Holder continuous with respect to x, $\alpha(t)$ is continuous function on $[0,T]$. It is assumed that the functions $\psi_0(t)$, $\psi_1(t)$ are sufficiently smooth and satisfy the matching conditions $\psi_0(0)=\psi_0(T)$, $\psi_1(0)=\psi_1(T)$.

To the periodic boundary-value problem (2.1)-(2.3) we apply the parametrization method. Suppose $\lambda(x)=u(0,x)$, and in the problem (2.1)-(2.3) we make a substitution $u(t,x)=\widetilde{u}(t,x)+\lambda(x)$, where

$\tilde{u}(t,x)$ is a new unknown function. Then the periodic boundary value problem (2.1) - (2.3) reduces to the following equivalent problem

$$\frac{\partial \tilde{u}}{\partial t} = a(t,x)[\frac{\partial^2 \tilde{u}}{\partial x^2} + \ddot{\lambda}(x)] + c(t,x)[\tilde{u}(t,x) + \lambda(x)] + \alpha(t)\int_0^t \tilde{u}(\tau,x)d\tau + \alpha(t)\lambda(x) + f(t,x), \quad (2.4)$$

$(t,x) \in \Omega$,

$$\tilde{u}(0,x) = 0, \qquad x \in [0,\omega], \quad (2.5)$$

$$\tilde{u}(t,0) + \lambda(0) = \psi_0(t), \qquad \tilde{u}(t,\omega) + \lambda(\omega) = \psi_1(t), \qquad t \in [0,T], \quad (2.6)$$

$$\tilde{u}(T,x) = 0, \qquad x \in [0,\omega]. \quad (2.7)$$

From the conditions (2.5) and (2.6) follows $\lambda(0) = \psi_0(0)$, $\lambda(\omega) = \psi_1(0)$.

The problem (2.4) - (2.7) is an initial-boundary value problem for parabolic integro-differential equation with a parameter. An algorithm for finding the solution of problem (2.4) - (2.7) is constructed, which consists of two stages.

The first stage of the algorithm. We consider an auxiliary initial-boundary value problem for the parabolic integro-differential equation

$$\frac{\partial \tilde{u}}{\partial t} = a(t,x)\frac{\partial^2 \tilde{u}}{\partial x^2} + c(t,x)\tilde{u}(t,x) + \alpha(t)\int_0^t \tilde{u}(\tau,x)d\tau + \tilde{f}(t,x), \qquad (t,x) \in \Omega, \quad (2.8)$$

$$\tilde{u}(0,x) = 0, \qquad x \in [0,\omega], \ (2.9)$$

$$\tilde{u}(t,0) = \psi_0(t) - \psi_0(0), \qquad \tilde{u}(t,\omega) = \psi_1(t) - \psi_1(0), \qquad t \in [0,T], \quad (2.10)$$

where function $\tilde{f}(t,x)$ is continuous with respect to t, and is Holder continuous with respect to x.

The scheme of the polygonal method in relation to the problem (10.8) - (10.10). We take $h > 0$ and make a discretization by x: $x_i = ih$, $i = \overline{0,N}$, $Nh = \omega$, $\tilde{u}_i(t) = \tilde{u}(t,ih)$. The problem (2.8) - (2.10) is replaced by the following

$$\frac{d\tilde{u}_i}{dt} = a_i(t)\frac{\tilde{u}_{i+1} - 2\tilde{u}_i + \tilde{u}_{i-1}}{h^2} + c_i(t)\tilde{u}_i + \alpha(t)\int_0^t \tilde{u}_i(\tau)d\tau + \tilde{f}_i(t),\ \tilde{u}_i(0) = 0,\ i = \overline{0,N}, \quad (2.11)$$

$$\tilde{u}_0(t) = \psi_0(t) - \psi_0(0), \qquad \tilde{u}_N(t) = \psi_1(t) - \psi_1(0), \qquad t \in [0,T]. \quad (2.12)$$

Owing to linearity of system for $\forall h > 0$, there exists a unique solution of problem (2.11): $\{\tilde{u}_1(t),...,\tilde{u}_{N-1}(t)\}$ defined on $[0,T]$.

Taking the functions $\tilde{u}_{i+1}$, $\tilde{u}_{i-1}$ and the integral term to the right-hand side of every i th equation of the system (2.11), we applied the estimate from [15]:

$$\|\tilde{u}_i\| = \max_{t\in[0,T]}\{|\tilde{u}_i(t)|\} \le \frac{1}{2}\|\tilde{u}_{i-1}(t)\| + \frac{1}{2}\|\tilde{u}_{i+1}(t)\| + \frac{1}{2}\left\|\frac{\alpha(t)}{a_i(t)}\int_0^t \tilde{u}_i(\tau)d\tau\right\|h^2 + \frac{1}{2}\left\|\frac{\tilde{f}_i(t)}{a_i(t)}\right\|h^2.$$

Let $\xi_i = \|\tilde{u}_i\|$. Then, we obtain the following estimate

$$\xi_i \le \frac{1}{2}\xi_{i-1} + \frac{1}{2}\xi_{i+1} + \frac{1}{2}\left\|\frac{\alpha(t)}{a_i(t)}\right\|h^2 T\xi_i + \frac{1}{2}\left\|\frac{\tilde{f}_i(t)}{a_i(t)}\right\|h^2,\ i = \overline{1,N-1}. \quad (2.13)$$

Suppose that $\frac{1}{2}\max_{t\in[0,T]}\left\|\frac{\alpha(t)}{a_i(t)}\right\|Th^2 \le \chi < 1$, $i = \overline{1,N-1}$, from inequality (2.13) we get

$$\xi_i \le \frac{1}{2(1-\chi)}\xi_{i-1} + \frac{1}{2(1-\chi)}\xi_{i+1} + \frac{1}{2(1-\chi)}\left\|\frac{\tilde{f}_i(t)}{a_i(t)}\right\|h^2,\ i = \overline{1,N-1}. \quad (2.14)$$

Further, using sweep up and down, from (2.14) we have

$$\|\widetilde{u}_i\| \le \frac{N-i}{N(1-\chi)}\|\widetilde{\psi}_0\| + \frac{i}{N(1-\chi)}\|\widetilde{\psi}_1\| + \frac{N-i}{N(1-\chi)}\sum_{j=1}^{i}\left\| j\frac{\widetilde{f}_j(t)}{a_j(t)}\right\| h^2 +$$

$$+\frac{i}{N(1-\chi)}\sum_{j=i+1}^{N-1}\left\|(N-j)\frac{\widetilde{f}_j(t)}{a_j(t)}\right\| h^2 \le K_2,$$

where $\widetilde{\psi}_0(t) = \psi_0(t) - \psi_0(0)$, $\widetilde{\psi}_1(t) = \psi_1(t) - \psi_1(0)$.

From this inequality it follows the next assertion

Theorem 2.1. *Let*

a) the assumptions with respect to the data of problem (2.1)-(2.3) are fulfilled;

b) the inequality $\frac{1}{2}\max_{t\in[0,T]}\left\|\frac{\alpha(t)}{a_i(t)}\right\| Th^2 \le \chi < 1$ *is valid, where* $a_i(t) = a(t, ih)$, $i = \overline{0, N}$.

Then problem (2.8)—(2.10) has a unique classical solution $\widetilde{u}^*(t,x)$, *and for it the estimate holds:*

$$\max_{t\in[0,T]}|\widetilde{u}^*(t,x)| \le \frac{\omega - x}{\omega(1-\chi)}\max_{t\in[0,T]}|\psi_0(t) - \psi_0(0)| + \frac{x}{\omega(1-\chi)}\max_{t\in[0,T]}|\psi_1(t) - \psi_1(0)| +$$

$$+\frac{\omega - x}{\omega(1-\chi)}\int_0^x z\cdot\max_{t\in[0,T]}\left|\frac{\widetilde{f}(t,z)}{a(t,z)}\right| dz + \frac{x}{\omega(1-\chi)}\int_x^\omega (\omega - z)\cdot\max_{t\in[0,T]}\left|\frac{\widetilde{f}(t,z)}{a(t,z)}\right| dz.$$

Integrating equation (2.8) by variable t and accounting condition (2.10), we have

$$\widetilde{u}(t,x) = \int_0^t a(\tau,x)\frac{\partial^2 \widetilde{u}(\tau,x)}{\partial x^2}d\tau + \int_0^t a(\tau,x)d\tau\cdot\ddot{\lambda}(x) + \int_0^t c(\tau,x)\widetilde{u}(\tau,x)d\tau +$$

$$+\int_0^t c(\tau,x)d\tau\cdot\lambda(x) + \int_0^t \alpha(\tau)\int_0^\tau \widetilde{u}(\tau_1,x)d\tau_1 d\tau + \int_0^t \alpha(\tau)d\tau\cdot\lambda(x) + \int_0^t f(\tau,x)d\tau. \quad (2.15)$$

From expression (2.15), we determine the value of function $\widetilde{u}(t,x)$ for $t = T$ and replace in the condition (2.7):

$$\int_0^T a(\tau,x)d\tau\cdot\ddot{\lambda}(x) = -\left[\int_0^T c(\tau,x)d\tau + \int_0^T \alpha(\tau)d\tau\right]\cdot\lambda(x) -$$

$$-\int_0^T a(\tau,x)\frac{\partial^2 \widetilde{u}(\tau,x)}{\partial x^2}d\tau - \int_0^T c(\tau,x)\widetilde{u}(\tau,x)d\tau - \int_0^T \alpha(\tau)\int_0^\tau \widetilde{u}(\tau_1,x)d\tau_1 d\tau - \int_0^T f(\tau,x)d\tau. \quad (2.16)$$

The second stage of the algorithm. We consider the auxiliary problem

$$\int_0^T a(\tau,x)d\tau\cdot\ddot{\lambda}(x) = -\left[\int_0^T c(\tau,x)d\tau + \int_0^T \alpha(\tau)d\tau\right]\cdot\lambda(x) - g(x), \quad (2.17)$$

where $g(x)$ is continuous function on $[0,\omega]$.

From the matching condition we obtain

$$\lambda(0) = \psi_0(0),\ \lambda(\omega) = \psi_1(0). \quad (2.18)$$

The problem (2.17), (2.18) is the two-point boundary value problem for the second-order differential equation with respect to a function $\lambda(x)$.

Let $\widetilde{a}_1(x) = \int_0^T a(\tau,x)d\tau$, $\widetilde{b}_1(x) = \int_0^T c(\tau,x)d\tau + \int_0^T \alpha(\tau)d\tau$, The following assertion given of a condtions unique solvability to problem (2.17), (2.18).

Theorem 2.2. *Let*

a) the assumptions with respect to the data of problem (2.1)-(2.3) are fulfilled;

b) the condition $\int_0^T a(\tau, x)d\tau \neq 0$ *is valid for all* $x \in [0, \omega]$;

c) the inequality holds: $\max\left(1, \frac{2}{\omega}\right)\left[e^{\alpha\omega} - 1 - \alpha\omega\right] < 1$, *where* $\alpha = \max\left(1, 1/\max_{x\in[0,\omega]} |\tilde{a}_1(x)|\right) \cdot \max\left(\max_{x\in[0,\omega]} |b_1(x)|, 1\right)$.

Then problem (2.17), (2.18) has a unique solution $\lambda^*(x)$.

Theorems 2.1 and 2.2 given of the conditions unique solvability of auxiliary problems (2.8)-(2.10) and (2.17), (2.18) in the terms of initial data.

On each step of the algorithm: 1) the initial-boundary value problem for the parabolic integro-differential equation is solved at a fixed $\lambda(x)$; 2) a two-point boundary value problem for a second-order differential equation is solved at a fixed $\tilde{u}(t, x)$.

Conditions of Theorems 2.1 and 2.2 guarantee of realizable and convergence of proposed algorithm.

The proof of the convergence of the proposed algorithms is based on the results of the work [16-23].

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СЫНЫҚТАР ӘДІСІНІҢ ЖҮКТЕЛГЕН ЖӘНЕ ИНТЕГРАЛДЫҚ-ДИФФЕРЕНЦИАЛДЫҚ ПАРАБОЛАЛЫҚ ТЕҢДЕУЛЕР ҮШІН ПЕРИОДТЫ ЕСЕПТІ ШЕШУГЕ ҚОЛДАНЫЛУЫ

Аннотация. Бірінші бөлімде жүктелген параболалық теңдеу үшін периодты есеп тіктөртбұрышты облыста қарастырылады. Сынықтар әдісін пайдалана отырып біз жүктелген параболалық теңдеулер үшін периодты есепті шешудің алгоритмдерін құрамыз. Алгоритмнің жинақтылығы дәлелденеді. Зерттеліп отырған есептің шешішілімділік шарттары бастапқы берілімдер терминдерінде берілген. Екінші бөлімде интегралдық-дифференциалдық параболалық теңдеу үшін периодты есеп тіктөртбұрышты облыста қарастырылады. Сынықтар әдісі интегралдық-дифференциалдық параболалық теңдеулерге дамытылады. Интегралдық-дифференциалдық параболалық теңдеулер үшін периодты есептің шешімін табу алгоритмдері құрылған және олардың жинақтылығы дәлелденген. Зерттеліп отырған есептің шешішілімділік шарттары бастапқы берілімдер терминдерінде берілген.

Тірек сөздер: периодты есеп, жүктелген параболалық теңдеулер, интегралдық-дифференциалдық параболалық теңдеулер, алгоритм, бірмәнді шешілімділік.

А.Т. Асанова

ПРИМЕНЕНИЕ МЕТОДА ЛОМАНЫХ К РЕШЕНИЮ ПЕРИОДИЧЕСКОЙ ЗАДАЧИ ДЛЯ НАГРУЖЕННОГО И ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ

Аннотация. В первой части рассматривается периодическая задача для нагруженного параболического уравнения в прямоугольной области. Используя метод ломаных мы строим алгоритмы нахождения решения периодической краевой задачи для нагруженных параболических уравнений. Доказывается сходимость алгоритма. Условия разрешимости исследуемой задачи даются в терминах исходных данных. Во второй части рассматривается периодическая задача для интегро-дифференциального параболического уравнения в прямоугольной области. Метод ломаных развивается на интегро-дифференциальные параболические уравнения. Построены алгоритмы нахождения решения периодической краевой задачи для интегро-дифференциальных параболических уравнений и доказана их сходимость. Условия разрешимости исследуемой задачи даются в терминах исходных данных.

Ключевые слова: периодическая задача, нагруженные параболические уравнения, интегро-дифференциальные параболические уравнения, метод ломаных, алгоритм, однозначная разрешимость.