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ON PROJECTIONAL ORTHOGONAL BASIS OF A LINEAR NON-SELF -ADJOINT OPERATOR

Abstract. In this paper we study spectral properties of a linear non-self-adjoint operator with an internal symmetry of the form:

$$PL = L^*P, \qquad LQ = QL^*;$$

where $P^* = P,\ Q^* = Q$ are orthogonal projections, L^* is an operator, adjoint to the operator L in the Hilbert space H. It is shown that a spectrum of such operator is real. In the case of a discrete operator, with a complete system of eigenvectors and associated vectors, the projections of eigenvalues and associated vectors of the operator L and its adjoint operator form an orthonormal basis. A class of Sturm-Liouville operators with such symmetry is found, moreover, it is found that the characteristic function of such an operator factorizes. An illustrative example is provided.

Keywords: Linear non-self-adjoint operator, real spectrum, basis, root vectors, completeness, theory of electric signals, plasma theory, discrete operator, invariant subspaces, root subspaces, completely continuous operator, eigenvectors and associated vectors, internal symmetry, projection, resolvent.

1. Introduction. The aim of the paper is to study the spectral properties of a certain class of linear non-self-adjoint operators L with a real spectrum that have the following internal symmetry

2.

$$PL = L^*P, \qquad LQ = QL^*; P^* = P, \qquad Q^* = Q,$$

where L is a linear operator with a domain $D(A)$, that belongs to the Hilbert space H, and P and Q are orthogonal projections, defined in this space.

We will assume that the Hilbert space H is separable.

We consider a completely continuous operator T, acting in the separable Hilbert space H. We denote by R_λ the following operator

$$(T - \lambda I)^{-1}. \tag{1.1}$$

The set of points of the plane λ, for which the operator (1.1) is everywhere defined and bounded, is called the resolvent set, and its complement is called the spectrum of the operator T. It is known that the spectrum of the completely continuous operator T consists of at most a countable number of points

$$\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n, \dots,$$

which can have a limit point only at zero. If the space H is infinite-dimensional, then zero is always a point of the spectrum of the completely continuous operator. Each non-zero point of the spectrum λ_s of the completely continuous operator T corresponds to a finite-dimensional invariant subspace K_s, which is defined as the set of elements of f, canceled by some power operator $T - \lambda_s I$:

$$(T - \lambda_s I)^n f = 0 \tag{1.2}$$

The subspace K_s is called root subspace. Dimension of the root subspace, corresponding to the point of the spectrum λ_s, we will denote by m_s.

In each of the root subspace K_s, since it is finite dimensional, it is possible to choose a basis, in which the transformation is written by a Jordan matrix. This basis consists of several chains of equalities:

$$f_{11}, f_{21}, \dots,\ f_{R_1 1};\ f_{12}, f_{22}, \dots,\ f_{R_2 2}, \dots;\ f_{1q}, f_{2q}, \dots,\ f_{R_q q}. \tag{1.3}$$

Each chain forms a basis in one of invariant subspaces on which this root subspace K_s splits. For elements of each chain the following equalities hold:

$$Tf_{1\tau} = \lambda_s f_{1\tau},\ Tf_{2\tau} = \lambda_s f_{2\tau} + f_{1\tau}, \dots,\ \ Tf_{R_\tau \tau} = \lambda_s f_{R_\tau \tau} + f_{R_\tau \tau}$$

The first element of the chain is an eigen element of the operator T, corresponding to the eigenvalue λ_s, and the others are so-called associated elements.

In the paper we consider linear non-self-adjoint operators, acting in the separable Hilbert space H and with a discrete spectrum. The last one means that all points of a spectrum of the operator A (with the possible exception of one) are isolated, and the corresponding them subspaces are finite-dimensional. A finite-dimensional invariant subspace of the operator A, concerning to a certain point of the spectrum λ_s, is usually called the root subspace. We will denote it by K_s.

A root subspace K_s can be characterized as a collection of elements f, which satisfy the following equation at some integer $m \geq 1$

$$(A - \lambda_s I)^m f = 0. \tag{0.1}$$

As is well known, completely continuous operators, as well as unbounded (for example, differential) operators that have completely continuous inverse, has a discrete spectrum.

The main problem of the paper is to study the conditions under which a system of finite-dimensional invariant (root) subspaces of an operator turns out to be a basis in H or in the range of the operator.

Definition 1.1. A system of elements $\{e_n\}, n = 1,2, \dots$ forms a basis in the space H, if any element $x \in H$ can be uniquely represented in the form of the convergent series

$$x = \sum_{n=1}^{\infty} x_n e_n.$$

Every basis is a complete uniformly minimal system. However, a complete minimal system may not form a basis in space. For example, the trigonometric system $e_0(t) = 1,\ e_{2n-1}(t) = \sin nt,\ e_{2n}(t) = \cos nt,\ (n = 1,2, \dots)$ is complete and uniformly minimal system in the space $C(-\pi, \pi)$, but does not form basis there.

Definition 1.2. System $\{e_i\},\ i = 1,2, \dots$ is called an unconditional basis in the space H if it remains a basis for any permutation of its elements.

Let T be a linear bounded operator, acting in the space H and have a bounded inverse. If the system $\{e_i\}$ is a basis, then the system $\{Te_i\}$ is also a basis. If $\{e_i\}$ is unconditional basis, then and $\{Te_i\}$ is unconditional basis.

In the Hilbert space H, any orthogonal basis is unconditional. It turns out that any unconditional basis in the Hilbert space can be represented in the form $\{Te_i\},\ \{e_i\}$ is an orthogonal normed basis. Such bases were called Riesz bases. They can be characterized by the following properties: there exist positive numbers m and M such that for any $x \in H$

$$m \cdot \sum_{i=1}^{\infty} |(x, e_i)|^2 \leq \|x\|^2 \leq M \cdot \sum_{i=1}^{\infty} |(x, e_i)|^2.$$

Completeness is a necessary condition to have a basis. We clarify that a system of finite-dimensional invariant subspaces of a certain operator is called complete in a Hilbert space H, if any element $h \in H$ can be approximated with predetermined accuracy by the norm of a finite linear combination of elements, each of which belongs to one of the invariant subspaces. It is well known that if some completely continuous operator is self-adjoint, then the system of its finite-dimensional invariant subspaces is complete in the range of values of the operator, moreover it forms an orthonormal (after normalization) basis (in this case the root subspaces turn out to be proper).

In the case of a general completely continuous operator, the completeness may not occur. The simplest example of this kind is the integration operator

$$Af = \int_0^x f(t)dt\,,\ 0 \le x \le 1, \tag{0.2}$$

which acts in the Hilbert system of functions, that are Lebesgue integrable square in the interval $[0,1]$. Further we will denote this space as $L_2(0,1)$. It is easy to verify that the operator (0.2), being completely continuous, has only a single point of the spectrum — zero and does not have any eigenvector. Consequently, it has no finite-dimensional invariant subspaces at all.

In the theory of non-self-adjoint operators in a Hilbert space, questions on completeness and basicity of systems of root vectors play an important role. For many classes of non-self-adjoint operators, completeness of system of root vectors has now been studied quite fully. Important results in this direction are contained in [2] - [11] and monographs [1].

Problems of basicity of root vector systems are investigated much less thoroughly than the questions of completeness. The basis condition of the root vector system was studied for dissipative operators by B.R. Mukminov [12], I.M. Glazman [13] and A.S. Markus [14], and for weakly perturbed self-adjoint and normal operators by A.S. Markus [15] and by Visitey and A.S. Markus [16]. The methods, developed in [12] - [17], make it possible to establish that a system of root vectors of an operator belongs only to the class of Bari bases [18]. The class of Bari bases is very narrow, and basis property of the system of root vectors has been established in [12] - [17] with rather strict restrictions on operator.

In [19], a completely new analytical method was proposed for questions on basicity, based on systematic use of theorems on interpolation by analytic functions. In this case, L. Carleson theorem [20] on interpolation by bounded analytic functions was used. In this paper, series of theorems are established, these theorems contain conditions, sufficient and, in some cases necessary, for a system of root subspaces of an operator in a Hilbert space to be Riesz basis in its closed linear hull. Connection of operator-theoretic and differential-theoretic considerations was made on basis of the well-known J.fon Neumann theorem [21] - [22]. Ideas and methods of this paper were continued in the monograph [23].

V.P. Mikhailov [24] and G.M. Keselman [25] proved the Riesz basis property of systems of eigenfunctions and associated functions of an ordinary n-th order differential operator with strongly regular boundary value conditions in L_2. In the case of irregular boundary value conditions, a system of eigenfunctions and associated functions of the problem does not form even usual basis in L_2.

Basicity problem was completely solved only for the Sturm-Liouville model operator

$$Ly = -y''(x) = \lambda^2 y(x), \qquad x \in (0,1),$$

$$a_{11}y(0) + a_{12}y'(0) + a_{13}y(1) + a_{14}y'(1) = 0,$$

$$a_{21}y(0) + a_{22}y'(0) + a_{23}y(1) + a_{24}y'(1) = 0,$$

in [26].

In the case when the boundary value conditions are regular, but not strongly regular, the question on basis property of systems of eigenfunctions and associated functions has not yet been completely solved, thus, very active research is being conducted in this direction [27] - [31].

Theory of bases has, besides the theoretical value, purely practical value, and is used in the theory of electrical signals [32] and plasma [33].

2. Research Methods.

Lemma 2.1. If a system of vectors $\{y_n\}$, $n = 1,2,...$ is complete in the space H, then systems $\{Py_n\}$ and $\{Qy_n\}$ are complete in the subspaces $H_1 = PH$ and $H_2 = QH$, respectively, where P and Q are orthogonal projections given by the formulas

$$P = \frac{I+S}{2}, \qquad Q = \frac{I-S}{2},$$

and the operator S has the form $Su(x) = u(1-x)$, $\forall u(x) \in L^2(0,1) = H$.

Proof. Let for some element $g \in H_1$ of the subspace H_1 the following equality holds:

$$(g, Py_n) = 0, \qquad n = 1,2,...$$

where $(\cdot,\cdot)$ is a dot product in the space H, then there exists an element of the space H, such that $g = Pf$, consequently,

$$(Pf, Py_n) = 0, \qquad (P^2 f, y_n) = (Pf, y_n) = (g, y_n) = 0, \qquad n = 1,2,...$$

due to completeness of the system $\{y_n\}$ in the space H we have $g = 0$, that is required to prove.

Example.

It is known that the system $\{e^{inx}\}, n = 0, \pm1, \pm2, ...$ is complete in the space $H = L^2(-\pi,\pi)$. We put that $Su(x) = u(1-x)$ and

$$P = \frac{I+S}{2}, \qquad Q = \frac{I-S}{2},$$

then the system $u_n(x) = Pe^{inx} = \frac{e^{inx}+e^{-inx}}{2} = \cos nx$ is complete in the space of even functions $H_1 = PH$; The system $v_n(x) = ae^{inx} = i \sin nx$ is complete in the subspace of old functions $H_2 = QH$.

Let an operator L be densely defined and have a completely continuous inverse operator L^{-1}. Then a spectrum of the operator L is discrete and consists of only eigenvalues. Suppose that the following formulas hold:

$$PL = L^*P, \qquad LQ = QL^*,$$

where P and Q are orthogonal projections, i.e.

$$P^* = P, \qquad Q^* = Q.$$

Lemma 2.2. If y_n is an eigenvector, and $\widetilde{y_n}$ is an associated vector of the operator L, corresponding to the eigenvalue λ_n and the equality $Py_n = 0$ holds, then $P\widetilde{y_n} \neq 0$ and the vector $P\widetilde{y_n}$ is eigenvector for the adjoint operator L^*.

Proof. By assumption of the theorem, we have $L\widetilde{y_n} - \lambda_n \widetilde{y_n} = K_n \cdot y_n$, where K_n is some nonzero constant, then

$$PL\widetilde{y_n} - \lambda_n P\widetilde{y_n} = K_n P y_n = 0, => L^* P\widetilde{y_n} - \lambda_n P\widetilde{y_n} = 0;$$

If $P\widetilde{y_n} = 0$, then $\widetilde{y_n} = (P+Q)\widetilde{y_n} = Q\widetilde{y_n}$, $=>$ $LQ\widetilde{y_n} - \lambda_n Q\widetilde{y_n} = K_n \cdot y_n$.

Acting by the operator Q to the both sides of this equality, and taking into account $Qy_n \neq 0$ (otherwise we have $y_n = 0$), we get

$$QLQ\widetilde{y_n} - \lambda_n Q\widetilde{y_n} = K_n \cdot Qy_n \neq 0.$$

Due to the formula $LQ = QL^*$, we obtain

$$Q^2 L^* \widetilde{y_n} - \lambda_n Q\widetilde{y_n} = K_n \cdot Qy_n \neq 0, \qquad QL^* \widetilde{y_n} - \lambda_n Q\widetilde{y_n} = K_n \cdot Qy_n,$$

$$LQ\widetilde{y_n} - \lambda_n Q\widetilde{y_n} = K_n \cdot Qy_n \neq 0.$$

Thus, due to the formula, $\widetilde{y_n} = Q\widetilde{y_n}$, we get

$$LQQ\widetilde{y_n} - \lambda_n Q\widetilde{y_n} = K_n \cdot Qy_n \neq 0.$$

We obtained a contradiction, since the self-adjoint operator LQ has no adjoint vectors, therefore $P\widetilde{y_n} \neq 0$ and $L^*P\widetilde{y_n} = \lambda_n P\widetilde{y_n}$.

Remark. From the equality $Ly_n = \lambda_n y_n$, when $Py_n = 0$, we have

$$L\left(\underbrace{Py_n}_{0} + Qy_n\right) = \lambda_n\left(\underbrace{Py_n}_{0} + Qy_n\right), \Rightarrow LQy_n = \lambda_n Qy_n, \qquad LQ(Qy_n) = \lambda_n Qy_n \neq 0;$$

i.e. Qy_n is an eigenvector of the operator LQ.

Lemma 2.3. if to the eigen function y_n there corresponds a nonzero associated function $\widetilde{y_n}$, then $Py_n = 0$ and $P\widetilde{y_n} \neq 0$.

Proof. By condition of the theorem, we have

$$Ly_n = \lambda_n y_n, \qquad y_n \neq 0, \qquad L\widetilde{y_n} - \lambda_n\widetilde{y_n} = K_n \cdot y_n, \qquad K_n \neq 0.$$

Then acting by the operator P to both sides of the last formula, and using the formula $PL = L^*P$, we have

$$PL\widetilde{y_n} - \lambda_n P\widetilde{y_n} = K_n Py_n = 0, \Rightarrow L^*P\widetilde{y_n} - \lambda_n P\widetilde{y_n} = K_n Py_n, \Rightarrow$$

$$L^*P(|P\widetilde{y_n}) - \lambda_n P\widetilde{y_n} = K_n Py_n.$$

If $Py_n \neq 0$, then $P\widetilde{y_n} \neq 0$, and the self-adjoint operator L^*P has an associated vector, which is impossible, consequently, if to the eigenvector y_n there corresponds an associated vector $\widetilde{y_n}$, then $Py_n = 0$, and from the previous lemma it follows that $P\widetilde{y_n} \neq 0$ and the vector $P\widetilde{y_n}$ is eigenvector for the operator L^*.

These three lemmas form the basis of our method.

3. Research results.

Theorem 3.1. If root vectors of the operators L and L^* are complete in the space H, and

$$1)\ PL = L^*P;$$

$$2)\ LQ = QL^*;$$

$$3)\ P^2 = P,\ P^* = P;\ Q^2 = Q,\ Q^* = Q,$$

then spectrum of the operator L is real, and normed projections of root vectors of the operators L and L^* form an orthonormal basis in H, i.e.

$$Pf = \sum_{n=1}^{\infty}(Pf, P\varphi_n)\frac{P\varphi_n}{\|P\varphi_n\|}, \qquad Qf = \sum_{n=1}^{\infty}(Qf, Q\psi_n)\frac{Q\psi_n}{\|Q\psi_n\|},$$

where $\{\varphi_n\}$, $n = 1,2,\ldots$ are root vectors of the operator L, $\{\psi_n\}$, $n = 1,2,\ldots$ are root vectors of the operator L^*.

Proof. From the formulas 1) and 2), we have

$$(PL)^* = L^*P^* = L^*P = PL;\ (LQ)^* = Q^*L^* = QL^* = LQ;$$

consequently, operators PL and LQ are self-adjoint in the space H.

$$\text{If } Ly_n = \lambda_n y_n, \text{ then } PLy_n = \lambda_n Py_n,\ L^*Py_n = \lambda_n Py_n,\ L^*P(Py_n) = \lambda_n Py_n,$$

consequently, as $Py_n \neq 0$, λ_n is an eigenvalue of the self-adjoint operator, therefore it is real.

$$\text{If } Ly_n = \lambda_n y_n \text{ and } Py_n = 0, \text{ then } PLy_n = \lambda_n Py_n = 0,\ L(Py_n + Qy_n) = \lambda_n(P+Q)y_n,$$

$$LQy_n = \lambda_n Qy_n \text{ and } Qy_n \neq 0,$$

hence, and in this case λ_n is an eigenvalue of the self-adjoint operator LQ, thus it is a real value.

By our assumption system of eigen and associated functions $\{\varphi_n\}, n = 1,2, ...,$ of the operator L is complete in the space H, then the system $\{P\varphi_n\}, n = 1,2, ...,$ is complete in the subspace $H_1 = PH$ (see Lemma 2.1). Since all eigenvalues λ_n $(n = 1,2, ...)$ of the operator L are real, then spectra of the operators L and L^* are the same.

If $L\varphi_n = \lambda_n\varphi_n$, then due to the formula $PL = L^*P$ we have $PL\varphi_n = \lambda_n P\varphi_n$, $L^*P\varphi_n = \lambda_n P\varphi_n$, consequently, the vector $P\varphi_n$ is eigenvector for the self-adjoint operator L^*P.

If $P\varphi_n \neq 0$, then due to Lemma 2.3, there is no associated function.

If $P\varphi_n = 0$, then there may be an attached vector $\widetilde{\varphi_n}$, such that

$$L\widetilde{\varphi_n} - \lambda_n\widetilde{\varphi_n} = K_n\varphi_n, \qquad K_n \neq 0.$$

Then $PL\widetilde{\varphi_n} - \lambda_n P\widetilde{\varphi_n} = K_n P\varphi_n = 0$, moreover, due to Lemma 2.2, we have $P\widetilde{\varphi_n} \neq 0$.

Operator L does not have associated vectors higher than first order. Indeed, if

$$L\widetilde{\widetilde{\varphi_n}} - \lambda_n\widetilde{\widetilde{\varphi_n}} = K_n\widetilde{\varphi_n}, \qquad K_n \neq 0,$$

then

$$P\varphi_n = 0, \qquad P\widetilde{\varphi_n} \neq 0,$$

thus

$$PL\widetilde{\widetilde{\varphi_n}} - \lambda_n P\widetilde{\widetilde{\varphi_n}} = K_n P\widetilde{\varphi_n} \neq 0, \Rightarrow L^*P\widetilde{\widetilde{\varphi_n}} - \lambda_n P\widetilde{\widetilde{\varphi_n}} = K_n P\widetilde{\varphi_n} \neq 0, \Rightarrow$$

$$\Rightarrow L^*P\left(P\widetilde{\widetilde{\varphi_n}}\right) - \lambda_n P\widetilde{\widetilde{\varphi_n}} = K_n P\widetilde{\varphi_n} \neq 0.$$

Consequently, $P\widetilde{\widetilde{\varphi_n}} \neq 0$, and this contradicts self-adjointness of the operator L^*P.

Therefore, if the sequence $\{\varphi_n\}, n = 1,2, ...,$ consists of eigen and associated functions of the operator L, then the sequence $\{P\varphi_n\}, n = 1,2, ...$ consists of eigenvectors of the self-adjoint operator L^*P, hence it is a complete and orthogonal system. Rejecting zero elements, if there is any of them, we get a complete orthogonal system $\{P\varphi_n\}, n = 1,2, ...$ (cleaned system). Consequently, the system $\{P\varphi_n/\|P\varphi_n\|\}$, $n = 1,2, ...$ is an orthonormal basis of the space $H_1 = PH$, i.e. for any vector f from H the following decomposition holds

$$Pf = \sum_{n=1}^{\infty}(Pf, P\varphi_n)\frac{P\varphi_n}{\|P\varphi_n\|};$$

2) Let a system of eigen and associated functions $\{\psi_n\}, n = 1,2, ...,$ of the operator L^* be complete in the space H, then the system $\{Q\psi_n\}, n = 1,2, ...,$ is complete in the subspace $H_2 = QH$.

If $L^*\psi_n = \lambda_n\psi_n$, then $QL^*\psi_n = \lambda_n Q\psi_n$,

$LQ\psi_n = \lambda_n Q\psi_n$, $LQ(Q\psi_n) = \lambda_n Q\psi_n$;

If $Q\psi_n \neq 0$, then due to Lemma 2.3, there is not any associated vector. If $Q\psi_n = 0$, then maybe there is an attached vector $\widetilde{\psi_n}$, i.e.

$$L^*\widetilde{\psi_n} - \lambda_n\widetilde{\psi_n} = K_n\psi_n, \qquad K_n \neq 0,$$

then $QL^*\widetilde{\psi_n} - \lambda_n Q\widetilde{\psi_n} = K_n Q\psi_n = 0, \Rightarrow$

$$LQ\widetilde{\psi_n} - \lambda_n Q\widetilde{\psi_n} = 0, \Rightarrow LQ\left(Q\widetilde{\psi_n}\right) - \lambda_n Q\widetilde{\psi_n} = 0,$$

moreover, due to Lemma 2.2, we have $Q\widetilde{\psi_n} \neq 0$.

Consequently, in any case the vector $Q\widetilde{\psi_n}$ is eigenvector for the operator LQ. Due to self-adjointness of the operator LQ, eigenvectors $\{Q\psi_n\}$ are mutually orthogonal, and according to our assumption and Lemma 2.1, are complete in the subspace $H_2 = QH$, consequently, the system $Q\psi_n/\|Q\psi_n\|$, $n = 1,2,...$ forms a orthonormal basis in the subspace H_2, i.e.

$$Qf = \sum_{n=1}^{\infty}(Qf, Q\psi_n)\frac{Q\psi_n}{\|Q\psi_n\|}.$$

4. Discussion.

We consider the model Sturm - Liouville operator in the space $L^2(0,1)$.

$$Ly = -y''(x),\ x \in (0,1), \tag{4.1}$$

$$\begin{cases} a_{11}y(0) + a_{12}y'(0) + a_{13}y(1) + a_{14}y'(1) = 0, \\ a_{21}y(0) + a_{22}y'(0) + a_{23}y(1) + a_{24}y'(1) = 0, \end{cases} \tag{4.2}$$

where a_{ij} $(i = 1,2; j = 1,2,3,4) -$ are arbitrary complex numbers. By Δ_{ij} we denote minors of the boundary matrix:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

Therefore,

$$\Delta_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix},\ (i,j = 1,2,3,4). \tag{4.3}$$

If the following inequality holds

$$\Delta = \Delta_{12} + \Delta_{13} + \Delta_{14} + \Delta_{32} + \Delta_{34} \neq 0,$$

then the operator (4.1) - (4.2) is invertible and its inverse operator L^{-1} is completely continuous.

Theorem 4.1. Invertible Sturm-Liouville operator satisfies the following equalities

$$PL = L^+P,\ LQ = QL^+, \tag{4.4}$$

if and only if it has the form

$$Ly = -y''(x),\ x \in (0,1), \tag{4.1}$$

$$\begin{cases} y(0) + ky'(0) + y(1) - ky'(1) = 0, \\ (1-2l)y(0) - (2\alpha + k)y'(0) - (1-2l)y(1) - (2l - 2\alpha - k)y'(1) = 0, \end{cases}$$ we denote as (4.5),

its adjoint has the form:

$$Lz = -z''(x),\ x \in (0,1) \tag{4.2$^+$}$$

$$\begin{cases} (1-2l)z(0) - lz'(0) - (1-2l)z(1) - lz'(1) = 0, \\ (1-k-2\bar{\alpha})z(0) - \bar{\alpha}z'(0) + (k + 2\bar{\alpha})z(1) - (k + \bar{\alpha})z'(1) = 0, \end{cases} \tag{4.5$^+$}$$

where $k = \bar{k},\ l = \bar{l} -$ is a real number, $\alpha -$ is an arbitrary complex number, P and Q are orthogonal projections given by the formulas:

$$P = \frac{I+S}{2},\ Q = \frac{I-S}{2}, \tag{4.6}$$

where $I -$ is a unit operator, operator S is defined by the formula

$$Su(x) = u(1-x), \ \forall u(x) \in L^2(0,1). \tag{4.7}$$

Theorem 4.2. If for the invertible Sturm-Liouville operator (4.1) - (4.2) the following formulas hold

a) $PL = L^+P$,
b) $LQ = QL^+$; (4.4)

where

$$P = \frac{I+S}{2}, \ Q = \frac{I-S}{2}, \tag{4.6}$$
$$Su(x) = u(1-x), \ \forall u(x) \in L^2(0,1), \tag{4.7}$$

then the characteristic function of this operator is factorized as follows:

a) If $kl(1-2l) \neq 0$, then

$$\Delta(\lambda) = 2l\left(\frac{2l-1}{l}\cdot\frac{\sin\frac{\lambda}{2}}{\lambda} - \cos\frac{\lambda}{2}\right)\cdot\left(kl\sin\frac{\lambda}{2} + \cos\frac{\lambda}{2}\right);$$

b) If $l = \frac{1}{2}$, then

$$\Delta(\lambda) = -\frac{\cos\frac{\lambda}{2}}{2}\left(\lambda k\cos\frac{\lambda}{2} + \cos\frac{\lambda}{2}\right);$$

c) If $l = 0, \ k \neq 0$, then

$$\Delta(\lambda) = -2\sin\frac{\lambda}{2}\left(k\sin\frac{\lambda}{2} + \frac{\cos\frac{\lambda}{2}}{2}\right);$$

d) If $l = 0, \ k = 0$, then

$$\Delta\Delta(\lambda) = -\frac{2\sin\frac{\lambda}{2}\cos\frac{\lambda}{2}}{2};$$

where $k, \ l -$ are real coefficients of the boundary conditions (4.5), and the characteristic function $\Delta(\lambda)$ has the form:

$$\Delta(\lambda) = \Delta_{12} + \Delta_{34} + \Delta_{13}\frac{\sin\lambda}{\lambda} + (\Delta_{14} + \Delta_{32})\cos\lambda + \Delta_{24}\lambda\sin\lambda,$$

where $\Delta_{\square\square}$ are minors from (4.3).

Theorem 4.3. Eigenvalues and eigenfunctions of the boundary value problem

$$Ly = -y''(x), \ x \in (0,1),$$

$$\begin{cases} y(0) + y(1) = 0, \\ y(0) - y(1) - 2\alpha[y'(0) - y'(1)] = 0, \end{cases}$$

consist of two series:

a) $\lambda_n^{(1)} = 2n\pi, \ y_n^{(1)} = K_n \sin 2n\pi x, \ n = 1,2,\dots;$

b) $\lambda_n^{(2)} = (2n+1)\pi,\ y_n^{(2)} = B_n\left[2\alpha\cos(2n\pi+\pi)x + \frac{\sin(2n+1)\pi x}{(2n+1)\pi}\right],\ n = 1,2,\dots;$

where $\alpha -$ is an arbitrary complex number, $K_n,\ B_n -$ are arbitrary constants.

In this case, the normalized system

$$\left\{Py_n^{(2)}, Qz_n^{(2)}\right\},$$

forms an orthonormal basis of the space $L^2(0,1)$, where

$Py_n^{(2)} = \frac{B_n}{(2n+1)\pi}\sin(2n\pi+\pi)x; n = 0,1,2,\dots;$

$Qz_n^{(2)} = (-1)^n K_n \sin 2n\pi x, n = 1,2,\dots$

Theorem 4.3$^+$. Eigenvalues and eigenfunctions of the boundary value problem

$$L^+z = -z''(x) = \mu^2 z(x),\ x \in (0,1),$$
$$\begin{cases} z(0) - z(1) = 0, \\ z(0) - \bar{\alpha}[z'(0) + z'(1)] = 0; \end{cases}$$

consist of two series:

$$\text{a) } \mu_n^{(1)} = 2n\pi + \pi,$$
$$z_n^{(1)}(x) = A_n\cos(2n+1)\pi\left(\frac{1}{2} - x\right);$$
$$\text{b) } \mu_n^{(2)} = 2n\pi,\ n = 1,2,\dots$$
$$z_n^{(2)}(x) = K_n\left[4\bar{\alpha}n\pi\cos 2n\pi\left(\frac{1}{2} - x\right) - \sin 2n\pi\left(\frac{1}{2} - x\right)\right],$$

where $\alpha -$ is an arbitrary complex number, and $K_n -$ are arbitrary constants.

Moreover, $Qz_n^{(1)} = 0$, $Qz_n^{(2)} = (-1)^n K_n \sin 2n\pi x$, which confirms results of the main Theorem 3.1.

We note that Sturm-Liouville operators of the class, that we studied, are reconstructed in a single spectrum [34].

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СЫЗЫҚТЫҚ СЫҢАРЛЫ ОПЕРАТОРДЫҢ ОРТОГОНӘЛДІ ПРОЕКЦИЯЛЫҚ БАЗИСІ ТУРАЛЫ

Аннотация. Бұл еңбекте, мынадай

$$PL = L^*P, \qquad LQ = QL^*$$

ішкі симметриясы бар сызықтық сыңарлы оператордың спектрәлдік қасиеттері зерттелді, мұндағы $P^* = P$, $Q^* = Q$ –ортогонәлді проекторлар, ал L^*- сыңар оператор. Әңгіме Гилберттің сепарабелді H кеңістігінде болып отыр.

Мұндай операторлардың спектрі накты сандар өсінде жататыны көрсетілді, сондай-ақ меншікті және олармен еншілес векторлар системасы кеңістікте толық дискретті операторлардың түпкі векторларының проекциялары мен оған сыңар оператордың түпкі векторларының проекциялары бірігіп, H кеңістігінде ортогонәл базис құрайтыны көрсетілді, әрине, ортанормаланған соң.

Штурм-Лиувилл операторларының ішінен осындай симметриясы бар операторлар класы бөліп алынды. Мұндай опраторлардың характеристикалық функциялары көбейткіштерге жіктелетіні дәлелденді. Теореманың мәні мысал арқылы айқындала түсті.

Түйін сөздер: Сызықтық сыңарлы оператор, нақты спектр, түпкі векторлар, толымдылық, электр сигналдарының теориясы, плазманың теориясы, дискретті оператор, ішкікеңістіктер, инвариантты кеңістіктер, түпкі кеңістіктер, меншікті және еншілес векторлар, ішкі симметрия, проектор, резөлвента.

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О ПРОЕКЦИОННО ОРТОГОНАЛЬНОМ БАЗИСЕ ЛИНЕЙНОГО НЕСАМОСОПРЯЖЕННОГО ОПЕРАТОРА

Аннотация. В настоящей работе исследованы спектральные свойства линейного несамосопряженного оператора обладающего внутренней симметрией вида

$$L = L^*P, \qquad LQ = QL^*;$$

где $P^* = P,\ Q^* = Q$ −ортогональные проекторы, L^* − оператор, сопряженный к оператору L в гильбертовом пространстве H. Показан, что спектр такого оператора вещественный. В случае дискретного оператора, с полной системой собственных и присоединенных векторов, проекций собственных и присоединенных векторов оператора L и его сопряженного образуют ортонормированный базис. Найден класс операторов Штурма – Лиувилля, обладающий такой симметрией, при этом обнаружено, что характеристическая функция такого оператора факторизуется. Приведен иллюстративный пример.

Ключевые слова: Линейный несамосопряженный оператор, вещественный спектр, базис, корневые векторы, полнота, теория электрических сигналов, теория плазмы, дискретный оператор, инвариантные подпространства, корневые подпространства, вполне непрерывный оператор, собственные и присоединенные векторы, внутренняя симметрия, проектор, резольвента.

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