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ANALYTICAL EXPRESSIONS OF THE PERTURBING FUNCTIONS IN TWO PLANETARY THREE- BODY PROBLEM WITH MASSES VARYNG NON-ISOTROPICALLY WHEN AVAILABLE FOR REACTIVE FORCES

Abstract. The paper considers a two-planetary exoplanetary system with variable masses in the absolute coordinate system. The equations of motion are described with the Meshchersky equations. The masses of the parent star and the planets are considered variable, varying at different rates. The general case is investigated when the masses of bodies change with time anisotropically, at different rates. As a consequence of an anisotropic change in mass, reactive forces appear that significantly affect the dynamics of the exoplanetary system at the non-stationary stage of its evolution. The equations of motion have no integral, so the problem is investigated by perturbation theory methods developed for such non-stationary systems. The initial equations for the use of perturbation theory are the equations of motion in a relative coordinate system with the origin at the center of the parent star with mass. The methods of perturbation theory are used based on aperiodic motion along a quasiconic section. The motion of two planets, within the framework of the problem of three point bodies with variable masses varying anisotropically in the presence of reactive forces, are described by the equations of perturbed motion in the form of the Lagrange equation. Perturbating functions are expressed through the osculating elements of two planets. Analytical expressions for the expansion of perturbing functions into a series are obtained. The work highlights the main and indirect part of the perturbing functions is singled out. Exactly to the square of the eccentricities of the planets, actual decompositions are performed. The derived formulas allow us to study the evolution of orbital elements due to the variability of the masses of the parent star and planets. They allow us to describe dynamic effects in the two-planetary three-body problem with variable masses as a single planetary system at the non-stationary stage of its evolution. To perform complex analytical calculations, the Mathematica software package was used.

Key words: three-body problem with variable masses, non-stationary exoplanet systems, stars with variable masses, aperiodic motion, protoplanetary disk.

1. Introduction. Our solar system is considered a fairly evolved system (4.5 billion years). All orbits of the solar system are close to circular orbits $e \approx 0$, except for the orbit of Mercury. Mercury has eccentricity $e \approx 0.2$ the inclination of the plane of the orbit is 7 degrees. The great planets of the solar system, which is particularly well seen in the example of Saturn, "freezing" of the orbits has already occurred. Well-evolved systems have gone through so-called "freezing" of orbits (stable orbits are concentrated in one plane near the plane of the equator of the star).

Many exoplanetary systems (currently there are more than 4000 of them), we observe large variations in the slopes of the orbital planes to the equator of the star, which may indicate different evolutionary tracks of such systems. There is a known one exoplanet that moves in the opposite direction from the direction of the orbital motion (retrograde orbit) - an exoplanet called WASP-17b, located in the constellation Scorpio [2]. According to statistical analysis, it is known that the number of planets in specific exoplanetary systems varies from one to seven planets. In the exoplanetary system TRAPPIST-1, seven planets were discovered [3]. The stars of the spectral class G, which includes our Sun, have the largest number of exoplanetary systems. Venus is the only planet whose own rotation does not coincide with the direction of rotation of other planets in the solar system. This suggests that, because of the diversity of exoplanetary systems, there is a need for detailed studies of their dynamic evolution, especially at their non-stationary stages.

2. Research methods. Consider an exoplanetary system consisting of three intergravitating spherical celestial bodies with changing masses. Let, $m_0 = m_0(t)$ - the central parent star, $m_1 = m_1(t)$ - the inner planet and $m_2 = m_2(t)$ - the outer planet with variable masses. The motion of two planets, in the framework of the problem of three spherical bodies (which interact as material points) with variable masses changing anisotropically, in the presence of reactive forces, in an absolute coordinate system, are described by Meshchersky's equations [4]. The problem will be considered in a relative coordinate system with the origin at the center, the parent star, with mass $m_0 = m_0(t)$. Masses of bodies vary in different rates

$$\frac{\dot{m}_0}{m_0} \neq \frac{\dot{m}_1}{m_1} \neq \frac{\dot{m}_2}{m_2} \tag{2.1}$$

anisotropically. The problem is complicated, so we will investigate the problem using perturbation theory methods based on aperiodic motion along a quasiconical cross section [5]. It is expedient to proceed from the equation of motion in the relative coordinate system [4]. We will use the equations of perturbed motion in the form of the Lagrange equation. To write in explicit form the equation of perturbed motion in the form of the Lagrange equation, it is necessary to express through the osculating orbital elements of the perturbing functions for two planets.

The equations of the perturbed motion of two planets in a relative coordinate system will be written in the form [4]

$$\ddot{\vec{r}}_1 + f(m_0 + m_1)\frac{\vec{r}_1}{r_1^3} + \frac{\ddot{\gamma}_1}{\gamma_1}\vec{r}_1 = grad_{\vec{r}_1}\tilde{W}_1, \tag{2.2}$$

$$\ddot{\vec{r}}_2 + f(m_0 + m_2)\frac{\vec{r}_2}{r_2^3} + \frac{\ddot{\gamma}_2}{\gamma_2}\vec{r}_2 = grad_{\vec{r}_2}\tilde{W}_2, \tag{2.3}$$

$$\tilde{W}_1 = \tilde{U}_1 + F_1 + P_1, \tag{2.4}$$

$$F_1 = F_{1x}x_1 + F_{1y}x_1 + F_{1z}z_1, \quad P_1 = \frac{\ddot{\gamma}_1}{2\gamma_1}r_1^2, \tag{2.5}$$

$$\mu_2 = fm_2, \; r_{12} = |\vec{r}_2 - \vec{r}_1|, \; \gamma_1 = \frac{m_0(t_0) + m_1(t_0)}{m_0(t) + m_1(t)} = \gamma_1(t), \tag{2.6}$$

$$\tilde{W}_2 = \tilde{U}_2 + F_2 + P_2, \tag{2.7}$$

$$F_2 = F_{2x}x_2 + F_{2y}x_2 + F_{2z}z_2, \quad P_2 = \frac{\ddot{\gamma}_2}{2\gamma_2}r_2^2, \tag{2.8}$$

$$\mu_1 = fm_1, \; r_{21} = |\vec{r}_1 - \vec{r}_2|, \; \gamma_2 = \frac{m_0(t_0) + m_2(t_0)}{m_0(t) + m_2(t)} = \gamma_2(t), \tag{2.9}$$

$\tilde{U}_1$, $\tilde{U}_2$ - force functions of the Newtonian interaction of bodies, and we will assume that $r_1 < r_2$.

We express all the terms of the perturbing functions, through the orbital elements of the unperturbed motion. Out of them, the most complex is the expansion in a series of force functions of Newtonian interaction of bodies. It is expedient to distinguish the principal and the indirect part of the perturbing functions

$$\tilde{U}_1 = \frac{\mu_2}{\gamma_2 a_2} U_{1гл} - \frac{\mu_2}{\gamma_2 a_2} \alpha U_{1косв} \quad (2.10)$$

,

$$\tilde{U}_2 = \frac{\mu_1}{\gamma_2 a_2} U_{2гл} - \frac{\mu_1}{\gamma_2 a_2} \frac{1}{\alpha^2} U_{2косв}, \quad (2.11)$$

$$\alpha = \alpha(t) = \frac{\gamma_1 a_1}{\gamma_2 a_2} < 1, \quad (2.12)$$

$$\tilde{U}_{1гл} = \frac{\gamma_2 a_2}{r_{12}} = \gamma_2 a_2 \left(\frac{1}{r_{12}}\right), \ \tilde{U}_{1косв} = \left(\frac{r_1}{\gamma_1 a_1}\right)\left(\frac{\gamma_2 a_2}{r_2}\right)^2 \cos\psi, \quad (2.13)$$

$$\tilde{U}_{2гл} = \frac{\gamma_2 a_2}{r_{21}} = \gamma_2 a_2 \left(\frac{1}{r_{21}}\right), \ \tilde{U}_{2косв} = \left(\frac{r_2}{\gamma_2 a_2}\right)\left(\frac{\gamma_1 a_1}{r_1}\right)^2 \cos\psi. \quad (2.14)$$

Expressions in the right-hand parts of these formulas are expanded in a series along the osculating elements of aperiodic motion along a quasiconical section. The expansions of the perturbing functions (2.5), (2.8) are not particularly complicated, since the analytical expressions for the coordinates and the square of the modulus of the radius vector are simple [5]

$$x = \gamma\rho[\cos u \cdot \cos\Omega - \sin u \cdot \sin\Omega \cdot \cos i], \quad (2.15)$$

$$y = \gamma\rho[\cos u \cdot \sin\Omega + \sin u \cdot \cos\Omega \cdot \cos i], \quad (2.16)$$

$$z = \gamma\rho[\sin u \cdot \sin i], \quad (2.17)$$

$$r^2 = x^2 + y^2 + z^2 = \gamma^2\rho^2. \quad (2.18)$$

Their expansions in series are known [5,6]. Also, decompositions of the quantities $(r/\gamma a) = (\rho/a)$, $(\gamma a/r)^2 = (a/\rho)^2$ in the indirect part of the perturbing function (2.13), (2.14).

2.1 The decomposition of the perturbing function principal part. As noted above, the main difficulty lies in the expansion of the principal part of the perturbing function $\tilde{U}_{1гл}$, $\tilde{U}_{2гл}$. According to the vector $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$ follows

$$r_{12}^2 = r_2^2 - 2\vec{r}_1\vec{r}_2 + r_1^2 = r_2^2 - 2r_1r_2\cos\psi + r_1^2 \quad (2.19)$$

where, ψ - the angle between two radius-vectors.

We denote

$$\Delta^2 = r_{12}^2 = r_{21}^2 = r_2^2 - 2r_1r_2\cos\psi + r_1^2 \quad (2.20)$$

$$\Delta_0^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(u_1 - u_2), \quad (2.21)$$

$$\tilde{\Psi} = \cos\psi - \cos(u_1 - u_2), \quad (2.22)$$

where, $u_1 = \omega_1 + \theta_1$ и $u_2 = \omega_2 + \theta_2$ - the true longitudes of the inner and outer planets, respectively.

Then from (2.19) taking into account the notation (2.20)-(2.22) we obtain

$$\Delta^2=\Delta_0^2+\left(-2r_1r_2\tilde{\Psi}\right)=\Delta_0^2\left[1-\frac{2r_1r_2}{\Delta_0^2}\tilde{\Psi}\right]. \quad (2.23)$$

From the inequality (2.23) follows that

$$\frac{1}{\Delta}=\frac{1}{\Delta_0}\cdot\frac{1}{\sqrt{1-\frac{2r_1r_2}{\Delta_0^2}\tilde{\Psi}}}. \quad (2.24)$$

Using the well-known formula

$$\left(1-x\right)^{-1/2}=1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4+.....$$

let's expand the second factor in the right-hand side of (2.24). As a result, we get

$$\frac{1}{\Delta}=\frac{1}{\Delta_0}+r_1r_2\tilde{\Psi}\frac{1}{\Delta_0^3}+\frac{3}{2}\left(r_1r_2\tilde{\Psi}\right)^2\frac{1}{\Delta_0^5}+\frac{5}{2}\left(r_1r_2\tilde{\Psi}\right)^3\frac{1}{\Delta_0^7}+\ldots \quad (2.25)$$

Generalizing formula (2.25), we can write

$$\frac{1}{\Delta}=\frac{1}{\Delta_0}+r_1r_2\tilde{\Psi}\frac{1}{\Delta_0^3}+\frac{3}{2}\left(r_1r_2\tilde{\Psi}\right)^2\frac{1}{\Delta_0^5}+\frac{5}{2}\left(r_1r_2\tilde{\Psi}\right)^3\frac{1}{\Delta_0^7}+\ldots=\sum_{i=0}^{\infty}\frac{(2i)!}{(i!)^2}\cdot\left(\frac{1}{2}r_1r_2\tilde{\Psi}\right)^i\frac{1}{\Delta_0^{2i+1}}. \quad (2.26)$$

We express the right-hand side of (2.26) in terms of the orbital elements of the two planets. For this, it is necessary to express the quantities $\Delta_0^{-(2i+1)}$ and $\tilde{\Psi}$ through the orbital elements. Expressions in terms of orbital elements of quantities $r_1=\gamma_1\rho_1$, $r_2=\gamma_2\rho_2$, simple enough and known [8-10]. First, we obtain the necessary formulas for the expansion of the quantity determined by formula (2.22).

For the first summand of (2.22) we have

$$\cos\psi=\frac{x_1x_2+y_1y_2+z_1z_2}{r_1r_2}=\frac{x_1}{r_1}\cdot\frac{x_2}{r_2}+\frac{y_1}{r_1}\cdot\frac{y_2}{r_2}+\frac{z_1}{r_1}\cdot\frac{z_2}{r_2}=$$
$$=\left(\frac{x_1}{\gamma_1\rho_1}\right)\cdot\left(\frac{x_2}{\gamma_2\rho_2}\right)+\left(\frac{y_1}{\gamma_1\rho_1}\right)\cdot\left(\frac{y_2}{\gamma_2\rho_2}\right)+\left(\frac{z_1}{\gamma_1\rho_1}\right)\cdot\left(\frac{z_2}{\gamma_2\rho_2}\right) \quad (2.27)$$

Correspondingly, it follows from (2.15) - (2.17) that the coordinates of points can be written in the form

$$\left(\frac{x_1}{\gamma_1\rho_1}\right)\cdot\left(\frac{x_2}{\gamma_2\rho_2}\right)=\left[\{\cos u_1\}\cos\Omega_1-\{\sin u_1\}\sin\Omega_1\cos i_1\right]\cdot\left[\{\cos u_2\}\cos\Omega_2-\{\sin u_2\}\sin\Omega_2\cos i_2\right]$$
$$\left(\frac{y_1}{\gamma_1\rho_1}\right)\cdot\left(\frac{y_2}{\gamma_2\rho_2}\right)=\left[\{\cos u_1\}\sin\Omega_1+\{\sin u_1\}\cos\Omega_1\cos i_1\right]\cdot\left[\{\cos u_2\}\sin\Omega_2+\{\sin u_2\}\cos\Omega_2\cos i_2\right] \quad (2.28)$$
$$\left(\frac{z_1}{\gamma_1\rho_1}\right)\cdot\left(\frac{z_2}{\gamma_2\rho_2}\right)=\left[\{\sin u_1\}\sin i_1\right]\cdot\left[\{\sin u_2\}\sin i_2\right]$$

Formulas (2.27), (2.28) define the expression $\cos\psi$ in terms of orbital elements. In (2.22), it is still necessary to expand the second term in series

$$\begin{aligned}&\cos(u_1-u_2)=\cos u_1\cos u_2+\sin u_1\sin u_2=\\&=\cos(\omega_1+\theta_1)\cos(\omega_2+\theta_2)+\sin(\omega_1+\theta_1)\sin(\omega_2+\theta_2)=\\&=[\cos\omega_1\{\cos\theta_1\}-\sin\omega_1\{\sin\theta_1\}][\cos\omega_2\{\cos\theta_2\}-\sin\omega_2\{\sin\theta_2\}]+\\&+[\sin\omega_1\{\cos\theta_1\}+\cos\omega_1\{\sin\theta_1\}][\sin\omega_2\{\cos\theta_2\}+\cos\omega_2\{\sin\theta_2\}]\end{aligned}\quad(2.29)$$

The expression in curly brackets decomposes into an infinite series in powers of eccentricity [6]. As a result, we obtain an analytical expression $\tilde{\Psi}$ through the orbital elements of two planets.

The situation is more complicated in the expansion in a series of quantities $\Delta_0^{-(2i+1)}$. Equations (2.21) can be rewritten as

$$\begin{aligned}&\Delta_0^2=a_1^2\gamma_1^2\left(\frac{\rho_1}{a_1}\right)^2+a_2^2\gamma_2^2\left(\frac{\rho_2}{a_2}\right)^2-2a_1\gamma_1a_2\gamma_2\left(\frac{\rho_1}{a_1}\right)\left(\frac{\rho_2}{a_2}\right)\cos(u_1-u_2)=a_1^2\gamma_1^2(1+R_1)^2+\\&+a_2^2\gamma_2^2(1+R_2)^2-2a_1\gamma_1a_2\gamma_2(1+R_1)(1+R_2)\cos(u_1-u_2)=a_1^2\gamma_1^2+a_2^2\gamma_2^2-2a_1\gamma_1a_2\gamma_2\cos(u_1-u_2)+.\\&+a_1^2\gamma_1^2(2R_1+R_1^2)+a_2^2\gamma_2^2(2R_2+R_2^2)-2a_1\gamma_1a_2\gamma_2(R_2+R_1+R_1R_2)\cos(u_1-u_2)\end{aligned}\quad(2.30)$$

We denote by

$$\rho_0^2=\left[\gamma_1^2a_1^2+\gamma_2^2a_2^2-2\gamma_1\gamma_2a_1a_2\cos(u_1-u_2)\right],\ (2.31)$$

$$R_{12}=a_1^2\gamma_1^2(2R_1+R_1^2)+a_2^2\gamma_2^2(2R_2+R_2^2)-2a_1\gamma_1a_2\gamma_2(R_2+R_1+R_1R_2)\cos(u_1-u_2),$$

where, R_1, R_2 the remaining parts of the expansion of the radius-vector modules depend on the first and above the degree of eccentricity. Then it follows from (2.30) that

$$\Delta_0^2=\rho_0^2+R_{12}\ .\quad(2.32)$$

Therefore, we can write

$$\frac{1}{\Delta_0}=\frac{1}{\rho_0}\left(1+\frac{R_{12}}{\rho_0^2}\right)^{-1/2},\quad(2.33)$$

$$\begin{aligned}\frac{1}{\rho_0}&=\left[\gamma_1^2a_1^2+\gamma_2^2a_2^2-2\gamma_1\gamma_2a_1a_2\cos(u_1-u_2)\right]^{-1/2}=\\&=\frac{1}{\gamma_2a_2}\left[1+\alpha^2-2\alpha\cos(u_1-u_2)\right]^{-1/2}.\end{aligned}\quad(2.34)$$

We rewrite formula (2.33) in the form

$$\frac{1}{\Delta_0^{2i+1}}=\frac{1}{\rho_0^{2i+1}}\left(1+\frac{R_{12}}{\rho_0^2}\right)^{-(i+1/2)}.\quad(2.35)$$

Expanding the right-hand side of (2.35) in a Taylor series with ρ_0 respect to, we obtain

$$\frac{1}{\Delta_0^{2i+1}}=\frac{1}{\rho_0^{2i+1}}+(r_1-\gamma_1a_1)\frac{\partial}{\partial(\gamma_1a_1)}\left(\frac{1}{\rho_0^{2i+1}}\right)+(r_2-\gamma_2a_2)\frac{\partial}{\partial(\gamma_2a_2)}\left(\frac{1}{\rho_0^{2i+1}}\right)+\ldots\quad(2.36)$$

We denote by

$$\varepsilon_1 = \frac{r_1}{\gamma_1 a_1} - 1, \quad \varepsilon_2 = \frac{r_2}{\gamma_2 a_2} - 1. \tag{2.37}$$

From the known series expansion follows

$$\frac{r}{\gamma a} = \frac{\rho}{a} = 1 - e\cos M + \frac{e^2}{2}\left(1 - \cos 2M\right) + \frac{3e^3}{8}\left(\cos M - \cos 3M\right) + \\ + \frac{e^4}{3}\left(\cos 2M - \cos 4M\right) + O\left(e^5\right). \tag{2.38}$$

Therefore, ε_1 has order $O\left(e_1\right)$, where ε_2 has $O\left(e_2\right)$.

Let denote the $D_{m,n}$ differential operator

$$D_{m,n} = \left(\gamma_1 a_1\right)^m \left(\gamma_2 a_2\right)^n \frac{\partial^{m+n}}{\partial\left(\gamma_1 a_1\right)^m \partial\left(\gamma_2 a_2\right)^n}. \tag{2.39}$$

Then from (2.36) we obtain

$$\frac{1}{\Delta_0^{2i+1}} = \left[1 + \varepsilon_1 D_{1,0} + \varepsilon_2 D_{0,1} + \frac{1}{2!}\left(\varepsilon_1^2 D_{2,0} + 2\varepsilon_1\varepsilon_2 D_{1,1} + \varepsilon_2^2 D_{0,2}\right) + \ldots\right]\frac{1}{\rho_0^{2i+1}}. \tag{2.40}$$

However, from the relation (2.34) it follows that

$$\frac{1}{\rho_0^{2i+1}} = \left\{\frac{1}{\gamma_2 a_2}\left[1 - \alpha^2 - 2\alpha\cos\left(u_1 - u_2\right)\right]^{-1/2}\right\}^{(2i+1)} = \\ = \left(\gamma_2 a_2\right)^{-(2i+1)}\left[1 - \alpha^2 - 2\alpha\cos\left(u_1 - u_2\right)\right]^{-(i+1/2)} = \\ = \left(\gamma_2 a_2\right)^{-(2i+1)}\frac{1}{2}\sum_{j=-\infty}^{\infty} b_{i+1/2}^{(j)}\left(\alpha\right)\cos\left[j\left(u_1 - u_2\right)\right], \tag{2.41}$$

$$\frac{1}{2}b_s^{(j)}\left(\alpha\right) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos j\psi d\psi}{\left(1 - 2\alpha\cos\psi + \alpha^2\right)^s} \tag{2.42}$$

The quantities $b_s^{(j)}\left(\alpha\right)$ in the formula (2.42) are called the Laplace coefficients, each of which can be represented as a uniformly convergent series in α for all $\alpha < 1$ [7,6].

Let's denote

$$A_{i,j,m,n} = D_{m,n}\left(\left(a_2\gamma_2\right)^{-(2i+1)} b_{i+1/2}^{(j)}\left(\alpha\right)\right) = \\ \left(\gamma_1 a_1\right)^m\left(\gamma_2 a_2\right)^n \frac{\partial^{m+n}}{\partial\left(\gamma_1 a_1\right)^m \partial\left(\gamma_2 a_2\right)^n}\left(\left(a_2\gamma_2\right)^{-(2i+1)} b_{i+1/2}^{(j)}\left(\alpha\right)\right). \tag{2.43}$$

As a result, formula (2.40) has the form

$$\frac{1}{\Delta_0^{2i+1}} = \frac{1}{2}\sum_{j=-\infty}^{\infty}\left[A_{i,j,0,0} + \varepsilon_1 A_{i,j,1,0} + \varepsilon_2 A_{i,j,0,1} + \ldots\right]\cos j\left(u_1 - u_2\right). \tag{2.44}$$

if we generalize this expression, we get

$$\frac{1}{\Delta_0^{2i+1}}=\frac{1}{2}\sum_{j=-\infty}^{\infty}\left[\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{k=0}^{l}\binom{l}{k}\varepsilon_1^{k}\varepsilon_2^{l-k}A_{i,j,k,l-k}\right]\cos j\left(u_1-u_2\right). \qquad (2.45)$$

In calculating the partial derivatives with respect $A_{i,j,k,l-k}$ to $(\gamma_1 a_1)$ and $(\gamma_2 a_2)$ one should be careful, since $(\gamma_1 a_1)$ and $(\gamma_2 a_2)$ are also implicitly contained in the Laplace coefficients $b_{i+1/2}^{(j)}(\alpha)$.

Substituting (2.45) into (2.13), we have

$$U_{1гл}=\sum_{i=0}^{\infty}\frac{(2i)!}{(i!)^2}\cdot\left(\frac{\gamma_1\gamma_2}{2}\left(\frac{\rho_1}{a_1}\right)\left(\frac{\rho_2}{a_2}\right)\tilde{\Psi}\right)^i\frac{(\gamma_1 a_1)^i(\gamma_2 a_2)^{i+1}}{2}\times$$
$$\times\sum_{j=-\infty}^{\infty}\left[\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{k=0}^{l}\binom{l}{k}\varepsilon_{1k}\varepsilon_{2l-k}A_{i,j,k,l-k}\right]\cos j\left(u_1-u_2\right) \qquad (2.46)$$

We note that in the expression (2.46) the inclinations i_1 and i_2 are contained only in the quantity of $\tilde{\Psi}$.

2.2 The actual expansion of the perturbing functions to within second powers of small quantities. Let's consider the actual expansions of the perturbing functions to within second powers of small quantities. Up to second degrees of eccentricities we have [10]

$$r_1=\gamma_1\rho_1=\gamma_1 a_1\left(\frac{\rho_1}{a_1}\right)\approx\gamma_1 a_1\left[1+\frac{e_1^2}{2}+\left(-e_1\right)\cos M_1-\frac{e_1^2}{2}\cos 2M\right] \qquad (2.47)$$

$$r_2=\gamma_2\rho_2=\gamma_2 a_2\left(\frac{\rho_2}{a_2}\right)\approx\gamma_2 a_2\left[1+\frac{e_2^2}{2}+\left(-e_2\right)\cos M_2-\frac{e_2^2}{2}\cos 2M_2\right]. \qquad (2.48)$$

$$\sin\theta_1\approx\sin\lambda_1-(\Omega_1+\omega_1)+e_1\sin 2\lambda_1-(\Omega_1+\omega_1)+e_1^2\left(\frac{9}{8}\sin 3\lambda_1-(\Omega_1+\omega_1)-\frac{7}{8}\sin\lambda_1-(\Omega_1+\omega_1)\right)$$
$$\cos\theta_1\approx\cos\lambda_1-(\Omega_1+\omega_1)+e_1\left(\cos 2\lambda_1-(\Omega_1+\omega_1)-1\right)+e_1^2\left(\frac{9}{8}\cos 3\lambda_1-(\Omega_1+\omega_1)-\frac{7}{8}\cos\lambda_1-(\Omega_1+\omega_1)\right)$$
$$\sin\theta_2\approx\sin\lambda_2-(\Omega_2+\omega_2)+e_2\sin 2\lambda_2-(\Omega_2+\omega_2)+e_2^2\left(\frac{9}{8}\sin 3\lambda_2-(\Omega_2+\omega_2)-\frac{7}{8}\sin\lambda_2-(\Omega_2+\omega_2)\right) \qquad (2.49)$$
$$\cos\theta_2\approx\cos\lambda_2-(\Omega_2+\omega_2)+e_2\left(\cos 2\lambda_2-(\Omega_2+\omega_2)-1\right)+e_2^2\left(\frac{9}{8}\cos 3\lambda_2-(\Omega_2+\omega_2)-\frac{7}{8}\cos\lambda_2-(\Omega_2+\omega_2)\right),$$

where formula $M=\lambda-(\Omega+\omega)$ is being considered.

Using the above decomposition, we finally express the right-hand side of (2.22) in terms of the orbital elements.

Naturally, such cumbersome and complex analytical calculations are advisable to perform modern methods of computer algebra. We used the system of analytical computation Mathematica [11]

As a result, the right-hand side of (2.22) has the form

$$\tilde{\Psi}=\cos\psi-\cos(u_1-u_2)=$$

$$=\frac{1}{64}(-((-7\cos(\lambda_1-\omega_1-\Omega_1)+9\cos(3\lambda_1-\omega_1-\Omega_1)-7\sin(\lambda_1-\omega_1-\Omega_1)+$$

$$+9\sin(3\lambda_1-\omega_1-\Omega_1))e_1^2+8(\cos(-2\lambda_1+\omega_1+\Omega_1+1)+\sin(2\lambda_1-\omega_1-\Omega_1))e_1+8\cos(\lambda_1-\Omega_1))\times$$

$$\times((-7\cos(\lambda_2-\omega_2-\Omega_2)+9\cos(3\lambda_2-\omega_2-\Omega_2)-7\sin(\lambda_2-\omega_2-\Omega_2)+$$

$$+9\sin(3\lambda_2-\omega_2-\Omega_2))e_2^2+8(\cos(-2\lambda_2+\omega_2+\Omega_2+1)+\sin(2\lambda_2-\omega_2-\Omega_2))e_2+8\cos(\lambda_2-\Omega_2))+$$

$$+\sin(i_1)\sin(i_2)((-7\cos(\lambda_1-\omega_1-\Omega_1)+9\cos(3\lambda_1-\omega_1-\Omega_1)-7\sin(\lambda_1-\omega_1-\Omega_1)+$$

$$+9\sin(3\lambda_1-\omega_1-\Omega_1))e_1^2+8(\cos(-2\lambda_1+\omega_1+\Omega_1+1)+\sin(2\lambda_1-\omega_1-\Omega_1))e_1+8\sin(\lambda_1-\Omega_1))\times$$

$$\times((-7\cos(\lambda_2-\omega_2-\Omega_2)+9\cos(3\lambda_2-\omega_2-\Omega_2)-7\sin(\lambda_2-\omega_2-\Omega_2)+$$

$$+9\sin(3\lambda_2-\omega_2-\Omega_2))e_2^2+8(\cos(-2\lambda_2+\omega_2+\Omega_2+1)+\sin(2\lambda_2-\omega_2-\Omega_2))e_2+8\sin(\lambda_2-\Omega_2))-$$

$$-((-7\cos(\lambda_1-\omega_1-\Omega_1)+9\cos(3\lambda_1-\omega_1-\Omega_1)-7\sin(\lambda_1-\omega_1-\Omega_1)+$$

$$+9\sin(3\lambda_1-\omega_1-\Omega_1))e_1^2+8(\cos(-2\lambda_1+\omega_1+\Omega_1+1)+\sin(2\lambda_1-\omega_1-\Omega_1))e_1+8\sin(\lambda_1-\Omega_1))\times$$

$$\times((-7\cos(\lambda_2-\omega_2-\Omega_2)+9\cos(3\lambda_2-\omega_2-\Omega_2)-7\sin(\lambda_2-\omega_2-\Omega_2)+$$

$$+9\sin(3\lambda_2-\omega_2-\Omega_2))e_2^2+8(\cos(-2\lambda_2+\omega_2+\Omega_2+1)+\sin(2\lambda_2-\omega_2-\Omega_2))e_2+8\sin(\lambda_2-\Omega_2))+$$

$$+((-7\cos(\lambda_1-\omega_1-\Omega_1)+9\cos(3\lambda_1-\omega_1-\Omega_1)-7\sin(\lambda_1-\omega_1-\Omega_1)+$$

$$+9\sin(3\lambda_1-\omega_1-\Omega_1))(\cos(i_1)\cos(\Omega_1)+\sin(\Omega_1))e_1^2+8\left(\cos\left(\frac{1}{2}\right)+\sin\left(\frac{1}{2}\right)\right)\times$$

$$\times(\cos(i_1)\cos(\Omega_1)+\sin(\Omega_1))\left(\cos\left(-2\lambda_1+\omega_1+\Omega_1+\frac{1}{2}\right)-\sin\left(-2\lambda_1+\omega_1+\Omega_1+\frac{1}{2}\right)\right)e_1+$$

$$+8(\cos(i_1)\cos(\Omega_1)\sin(\lambda_1-\Omega_1)+\cos(\lambda_1-\Omega_1)\sin(\Omega_1)))((-7\cos(\lambda_2-\omega_2-\Omega_2)+$$

$$+9\cos(3\lambda_2-\omega_2-\Omega_2)-7\sin(\lambda_2-\omega_2-\Omega_2)+9\sin(3\lambda_2-\omega_2-\Omega_2))(\cos(i_2)\cos(\Omega_2)+\sin(\Omega_2))e_2^2+$$

$$+8\left(\cos\left(\frac{1}{2}\right)+\sin\left(\frac{1}{2}\right)\right)(\cos(i_2)\cos(\Omega_2)+\sin(\Omega_2))\left(\cos\left(-2\lambda_2+\omega_2+\Omega_2+\frac{1}{2}\right)-\sin\left(-2\lambda_2+\omega_2+\Omega_2+\frac{1}{2}\right)\right)e_2+$$

$$+8(\cos(i_2)\cos(\Omega_2)\sin(\lambda_2-\Omega_2)+\cos(\lambda_2-\Omega_2)\sin(\Omega_2)))+$$

$$+(-(7\cos(\lambda_1-\omega_1-\Omega_1)-9\cos(3\lambda_1-\omega_1-\Omega_1)+7\sin(\lambda_1-\omega_1-\Omega_1)-9\sin(3\lambda_1-\omega_1-\Omega_1))(\cos(\Omega_1)-\cos(i_1)\sin(\Omega_1))e_1^2+$$

$$+8\left(\cos\left(\frac{1}{2}\right)+\sin\left(\frac{1}{2}\right)\right)(\cos(\Omega_1)-\cos(i_1)\sin(\Omega_1))\left(\cos\left(-2\lambda_1+\omega_1+\Omega_1+\frac{1}{2}\right)-\sin\left(-2\lambda_1+\omega_1+\Omega_1+\frac{1}{2}\right)\right)e_1+$$

$$+8\cos(\lambda_1-\Omega_1)\cos(\Omega_1)-8\cos(i_1)\sin(\lambda_1-\Omega_1)\sin(\Omega_1))(-(7\cos(\lambda_2-\omega_2-\Omega_2)-9\cos(3\lambda_2-\omega_2-\Omega_2)+$$

$$+7\sin(\lambda_2-\omega_2-\Omega_2)-9\sin(3\lambda_2-\omega_2-\Omega_2))(\cos(\Omega_2)-\cos(i_2)\sin(\Omega_2))e_2^2+$$

$$+8\left(\cos\left(\frac{1}{2}\right)+\sin\left(\frac{1}{2}\right)\right)(\cos(\Omega_2)-\cos(i_2)\sin(\Omega_2))(\cos\left(-2\lambda_2+\omega_2+\Omega_2+\frac{1}{2}\right)- \qquad (2.50)$$

$$-\sin\left(-2\lambda_2+\omega_2+\Omega_2+\frac{1}{2}\right)\Bigg)e_2+8\cos(\lambda_2-\Omega_2)\cos(\Omega_2)-8\cos(i_2)\sin(\lambda_2-\Omega_2)\sin(\Omega_2))).$$

For the actual expansion of the quantities $\Delta_0^{-(2i+1)}$ from formula (2.44), we have

$$\frac{1}{\Delta_0}=\frac{1}{2}\sum_{j=-\infty}^{\infty}\left[A_{0,j,0,0}+\varepsilon_1 A_{0,j,1,0}+\varepsilon_2 A_{0,j,0,1}+\ldots\right]\cos j(u_1-u_2), \qquad (2.51)$$

$$\frac{1}{\Delta_0^3}=\frac{1}{2}\sum_{j=-\infty}^{\infty}\left[A_{1,j,0,0}+\varepsilon_1 A_{1,j,1,0}+\varepsilon_2 A_{1,j,0,1}+\ldots\right]\cos j\left(u_1-u_2\right), \tag{2.52}$$

$$\frac{1}{\Delta_0^5}=\frac{1}{2}\sum_{j=-\infty}^{\infty}\left[A_{2,j,0,0}+\varepsilon_1 A_{2,j,1,0}+\varepsilon_2 A_{2,j,0,1}+\ldots\right]\cos j\left(u_1-u_2\right), \tag{2.53}$$

where, according to (2.43), (2.42) are denoted by

$$A_{i,j,m,n}=D_{m,n}\left(\left(a_2\gamma_2\right)^{-(2i+1)}b_{i+1/2}^{(j)}(\alpha)\right)=$$

$$\left(\gamma_1 a_1\right)^m\left(\gamma_2 a_2\right)^n\frac{\partial^{m+n}}{\partial\left(\gamma_1 a_1\right)^m\partial\left(\gamma_2 a_2\right)^n}\left(\left(a_2\gamma_2\right)^{-(2i+1)}b_{i+1/2}^{(j)}(\alpha)\right), \tag{2.54}$$

$$\frac{1}{2}b_{i+1/2}^{(j)}(\alpha)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos j\psi d\psi}{\left(1-2\alpha\cos\psi+\alpha^2\right)^{i+1/2}}. \tag{2.55}$$

To obtain the expansion in an explicit form, up to a second order of small quantities, it suffices to retain the terms $j=-3,-2,-1,0,1,2,3$ in (2.51) - (2.53).

For example, consider $A_{1,-3,0,1}$. According to the formula (2.54), we can write

$$A_{1,-3,0,1}=D_{0.1}\left(\left(a_2\gamma_2\right)^{-3}b_{3/2}^{(-3)}(\alpha)\right)=\left(\gamma_2 a_2\right)\frac{\partial}{\partial\left(\gamma_2 a_2\right)}\left(\left(a_2\gamma_2\right)^{-3}b_{3/2}^{(-3)}(\alpha)\right)=-3\left(\gamma_2 a_2\right)^{-3}b_{3/2}^{(-3)}(\alpha)-$$
$$-\left(\gamma_1 a_1\right)\left(\gamma_2 a_2\right)^{-4}\frac{\partial}{\partial\alpha}\left[b_{3/2}^{-3}(\alpha)\right]=-3\left(\gamma_2 a_2\right)^{-3}b_{3/2}^{(-3)}(\alpha)-\left(\gamma_1 a_1\right)\left(\gamma_2 a_2\right)^{-4}\frac{d}{d\alpha}\left[b_{3/2}^{-3}(\alpha)\right]. \tag{2.56}$$

Similarly, other coefficients $A_{i,j,m,n}$ in formulas (2.51) - (2.53) are calculated.

3. Results. The performed analytical calculations lead us to the final results. In principle, we obtained the expansions of the perturbing functions up to any order with respect to small quantities.

Indeed, substituting the obtained analytical expressions (2.51) - (2.53), (2.47) - (2.48), (2.50) into formulas (2.25) and (2.13) we obtain the expansions of the principal part of the perturbing function. Obtained explicit formulas also give the possibility of writing in an analytic form the indirect parts of the perturbing functions. Also, using the obtained analytical expressions, we can write. As noted above, the expressions for the perturbing functions (2.5), (2.8) in terms of orbital elements are simple enough and follow from formulas (2.15) - (2.18).

Thus, the complete expressions of the perturbing function (2.4), (2.7) are expressed in terms of the orbital elements of the two planets.

Using the obtained formulas, the actual expansions of the perturbing functions are found to within a second order with respect to small quantities.

4. Discussion. In the paper, we examined two exoplanets in a relative coordinate system with the origin at the center of the parent star. The general formulas for the expansion in a series of perturbing functions in the two-planar three-body problem with masses varying anisotropically at various rates, based on aperiodic motion along a quasiconic section, are obtained for the first time. The relations obtained make it possible to expand the perturbing functions with any accuracy with respect to eccentricities and inclinations.

The results of this work open up new promising opportunities in the study of the dynamics of nonstationary gravitating systems. The obtained equations will be effectively used to study the dynamic evolution of exoplanetary systems because of an anisotropic change in the masses of the parent star and planets. In this case, the effects of the decrease in the mass of the parent star and the growth of the mass of the planets will be taken into account because of the accretion of matter from the remnants of the protoplanetary disc.

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РЕАКТИВТІ КҮШТІ ЕСЕПКЕ АЛЫП АНИЗАТРОПТЫ АЙНЫМАЛЫ МАССАДАҒЫ ЕКІ ПЛАНЕТАЛЫ ҮШ ДЕНЕ ЕСЕБІНІҢ ҰЙЫТҚУШЫ ФУНКЦИЯНЫҢ АНАЛИТИКАЛЫҚ ТЕҢДЕУЛЕРІ

Аннотация. Бұл жұмыста абсолютті координаталар жүйесіндегі айнымалы массалы екі планеталы экзопланеталық жүйе қарастырылған. Қозғалыс теңдеулері Мещерский теңдеулерімен сипатталады. Центрлік жұлдыздың және планеталардың массалары айнымалы әртүрлі қарқынмен өзгереді. Денелердің массалары уақыт бойынша анизотропты, әртүрлі қарқынмен өзгеретін жалпы жағдай зерттеледі. Экзопланеталы жүйе эволюциясының бейстационар сатысында массаның анизотропты өзгеруі айтарлықтай оның динамикасына әсерін тигізеді. Қозғалыс теңдеуінің интегралы болмағандықтан, бұл мәселе бейстационар жүйелерге өңделген ұйытқу теориясының әдістерімен зерттеледі. Ұйытқу теорисын қолдануға массалы центрлік жұлдыз салыстырмалы координаталар жүйесінің қозғалыс теңдеуінің басы ретінде қолданылады. Квазиконусты қима бойынша периодты емес қозғалыс негізінде ұйытқу теориясының әдістері қолданылады. Екі планетаның қозғалысы реактивті күшті ескергенде массалары айнымалы анизотропты өзгеретін үш нүкте дене есебінің шеңберінде ұйытқу теңдеуінің қозғалысы Лагранж теңдеулерінің формасында сипатталады. Ұйытқу функциялары екі планетаның оскуляцияланған элементтері арқылы өрнектеледі. Ұйытқу функцияларының қатарға жіктелінуінің аналитикалық теңдеулері алынды. Жұмыста ұйытқу функциясының басты және жанама бөлігі көрсетілді. Планеталардың эксцентриситеттерінің квадраттарына дейінгі дәлдіктегі нақты жіктеуі орындалды. Алынған формулалар центрлік жұлдыз және планеталардың массаларының айнымалылығына байланысты орбиталық элементтердің эволюциясын зерттеуге қолданылады. Қарастырылған екі планеталы үш дене есебінің бейстационар айнымалы эволюция сатысындағы динамикалық эффектілерді сипаттайды. Күрделі аналитикалық есептеулерді орындауда Mathematica пакет бағдарламасын қолдандық.

Түйін сөздер: айнымалы массалы үш дене есебі, бейстационар экзопланеталық жүйелер, айнымалы массалы жұлдыздар, апериодикалық қозғалыс, протопланеталық диск.

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АНАЛИТИЧЕСКИЕ ВЫРАЖЕНИЯ ВОЗМУЩАЮЩИХ ФУНКЦИИ В ДВУХПЛАНЕТНОЙ ЗАДАЧЕ ТРЕХ ТЕЛ С АНИЗАТРОПНО ИЗМЕНЯЮЩИМИСЯ МАССАМИ ПРИ НАЛИЧИИ РЕАКТИВНЫХ СИЛ

Аннотация. В работе рассматривается двухпланетная экзопланетная система с переменными массами в абсолютной системе координат. Уравнения движения описываются с уравнениями Мещерского. Массы родительской звезды и планет считается переменными, изменяющимися в различных темпах. Исследуется общий случай, когда массы тел меняются со временем анизотропно, в различных темпах. Как следствия анизотропного изменения масс появляются реактивные силы, которые существенно влияет на динамику экзопланетной системы на нестационарном этапе ее эволюции. Уравнения движения не имееют ни одного интеграла, поэтому проблема исследуются методами теории возмущении разработанных для таких нестационарных систем. Исходными для использования теории возмущений являются уравнения движения в относительной системе координат с началом в центре родительской звезды. Используется методы теории возмущении на базе апериодического движения по квазиконическому сечению. Движение двух планет, в рамках задачи трех точечных тел с переменными массами изменяющимися анизотропно при наличии реактивных сил, описываются уравнениями возмущенного движения в форме уравнения Лагранжа. Возмущающие функции выражаются через оскулирующие элементы двух планет. Получены аналитические выражения разложения в ряд возмущающих функции. В работе выделено главная и косвенная часть возмущающих функции. С точностью до квадрата эксцентриситетов планет выполнены фактические разложения. Найденные формулы позволяют исследовать эволюции орбитальных элементов из за переменности масс родительской звезды и планет. Они позволяют описывать динамических эффектов в рассматриваемой двухпланетной задаче трех тел с переменными массами как единая планетная система на нестационарном этапе ее эволюции. Для выполнения сложных аналитических вычислении использовали пакет программу Mathematica.

Ключевые слова: задача трех тел с переменными массами, нестационарные экзопланетные системы, звезды с переменными массами, апериодическое движения, протопланетный диск.

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