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ON THE MINIMALITY OF SYSTEMS OF ROOT FUNCTIONS OF THE LAPLACE OPERATOR IN THE PUNCTURED DOMAIN

Abstract. In this paper, we consider the Laplace operator in a punctured domain, which generates a class of “new” correctly solvable boundary value problems. And for this class of problems the resolvent formula is obtained. Also described are meromorphic functions that generate the root functions of the class of problems studied. The main goal is to study the minimality of root function systems. The paper is a continuation of [8], where a description is given of correctly solvable boundary value problems for the Laplace operator in punctured domains. The Laplace operator in the punctured domain, which generates the class of “new” correctly solvable boundary value problems, is considered, and the resolvent formula is obtained for the generated problems, and meromorphic functions are described that induce systems of functions. One of these systems is a system of eigenfunctions and associated functions. The last section is devoted to the study of the minimality of the system of root functions.

Key words: Laplace operator, punctured domain, resolvent, meromorphic function, correctly solvable boundary value problems, root functions system, minimal system.

1 Introduction

Operators of the form $L+K$, where $L=\Delta$ is the Laplace operator, and K is the operator of multiplication by a generalized function, appeared in the physical works of the 30s in connection with the problem of scattering neutral particles on the nucleus, when the interaction is strong at small distances and negligibly small to medium to large [1]. The model potential of such an interaction is the Dirac δ-function. A mathematical study of the operator $-\Delta+\mu\delta(x)$ was undertaken by Berezin and Fadeev [2], Minlos and Fadeev [3], Berezin [4]. In [2, 3], the operator $-\Delta+\mu\delta(x)$ was understood as an extension of the operator $L_0=-\Delta$ with the definition domain $D(L_0)=C_0^{\infty}(\mathbb{R}^3\backslash\{0\})$. Interesting are the works [5, 6], where the following important question was studied: if the operator $-\Delta+q$ with a singular potential q is already defined, can it be approximated in some sense by operators with smooth potentials so that the corresponding operators approximate the original in the sense of resolvent convergence. The paper [7] investigated the spectral properties of the Schrödinger operator with point interactions using positive definite functions. For complete review, see marked work and links to them.

The paper is a continuation of [8], where a description is given of correctly solvable boundary value problems for the Laplace operator in punctured domains. The Laplace operator in the punctured domain, which generates the class of “new” correctly solvable boundary value problems, is considered, and the resolvent formula is obtained for the generated problems, and meromorphic functions are described that induce systems of functions. One of these systems is a system of eigenfunctions and associated functions. The last section is devoted to the study of the minimality of the system of root functions.

Consider a differential expression

$$\Delta W(x,y) := \frac{\partial^2 W(x,y)}{\partial x^2} + \frac{\partial^2 W(x,y)}{\partial y^2}$$

in the punctured area $\Omega_0 = \Omega \backslash \{(x_0, y_0)\}$, here $\Omega = \{x^2 + y^2 < 1\}$ and (x_0, y_0) is internal fixed point of area Ω. We Turn from the expression $\Delta W(x,y)$ to the operator L_{σ_1} in the space $\mathbb{L}_2(\Omega)$.

Denote by $\mathcal{D}$ the set of all functions

$$h(x,y) = h_1(x,y) + \alpha\, G(x,y,x_0,y_0), (x,y) \in \Omega_0,$$

where $\alpha \in \mathbb{R}$, $h_1 \in \mathbb{D} = \{h_1 \in \mathbb{W}_2^2(\Omega), h_1|_{\partial\Omega} = 0\}$. Here and after $G(x,y,x_0,y_0)$ is Green function of the Dirichlet problem for the Laplace operator in Ω [9].

For $h(x,y) \in \mathcal{D}$ we introduce functionals

$$\alpha(h) = \frac{1}{2} \lim_{\delta \to +0} \left\{ \int_{y_0-\delta}^{y_0+\delta} \left[\frac{\partial h(x_0+\delta, y)}{\partial x} - \frac{\partial h(x_0-\delta, y)}{\partial x} \right] dy + \right.$$

$$\left. \int_{x_0-\delta}^{x_0+\delta} \left[\frac{\partial h(x, y_0+\delta)}{\partial y} - \frac{\partial h(x, y_0-\delta)}{\partial y} \right] dx \right\},$$

$$\beta(h) = \lim_{\delta \to +0} \int_{y_0-\delta}^{y_0+\delta} [h(x_0-\delta, y) - h(x_0+\delta, y)] dy,$$

$$\gamma(h) = \lim_{\delta \to +0} \int_{x_0-\delta}^{x_0+\delta} [h(x, y_0-\delta) - h(x, y_0+\delta)] dx.$$

Note that, the introduced functionals were first obtained in [8] to describe correctly solvable boundary value problems for the Laplace operator in the punctured domain Ω_0.

Consider in the space $\mathbb{L}_2(\Omega)$ the operator L_{σ_1} generated by the differential equation

(1.1) $$W(x,y) = f(x,y), \ (x,y) \in \Omega_0,$$

with external boundary condition

(1.2) $$W(x,y)|_{\partial\Omega} = 0,$$

and "internal boundary conditions"

(1.3) $$\frac{1}{2} \lim_{\delta \to +0} \int_{y_0-\delta}^{y_0+\delta} \left[\frac{\partial W(x_0+\delta, y)}{\partial x} - \frac{\partial W(x_0-\delta, y)}{\partial x} \right] dy +$$

$$+\frac{1}{2} \lim_{\delta \to +0} \int_{x_0-\delta}^{x_0+\delta} \left[\frac{\partial W(x, y_0+\delta)}{\partial y} - \frac{\partial W(x, y_0-\delta)}{\partial y} \right] dx - < \Delta W(x,y), \sigma_1(x,y) >= 0,$$

(1.4) $$\lim_{\delta \to +0} \int_{y_0-\delta}^{y_0+\delta} [W(x_0-\delta, y) - W(x_0+\delta, y)] dy = 0,$$

(1.5) $$\lim_{\delta \to +0} \int_{x_0-\delta}^{x_0+\delta} [W(x, y_0-\delta) - W(x, y_0+\delta)] dx = 0,$$

where $f(x,y), \sigma_1(x,y) \in \mathbb{L}_2(\Omega)$, $< f(x,y), \sigma_1(x,y) >$ mean inner product in $\mathbb{L}_2(\Omega)$.

2 Auxiliary statements

In the sequel, we will need a well-known statement from [10].

Theorem A. *Let* $h(x,y) \in \mathcal{D}$ *and there exists functional* $\alpha(h)$. *Then function*

$$W(x,y) = \iint\limits_{\Omega} G(x,y,\xi,\eta) f(\xi,\eta) d\xi d\eta + \int_{\partial\Omega} \frac{\partial G(x,y,\xi,\eta)}{\partial \overline{n}_{\xi\eta}} h(\xi,\eta) ds_{\xi,\eta} -$$

$$-\alpha(h) G(x,y,x_0,y_0) - \beta(h) \frac{\partial G(x,y,x_0,y_0)}{\partial \xi} - \gamma(h) \frac{\partial G(x,y,x_0,y_0)}{\partial \eta}$$

is uniqueness solution of the problem

(2.1) $$\Delta W(x,y) = f(x,y), (x,y) \in \Omega_0,$$

(2.2) $$W(x,y)|_{\partial\Omega} = h(x,y)|_{\partial\Omega},$$

(2.3) $$\frac{1}{2} \lim_{\delta \to +0} \int_{y_0-\delta}^{y_0+\delta} \left[\frac{\partial W(x_0+\delta,y)}{\partial x} - \frac{\partial W(x_0-\delta,y)}{\partial x}\right] dy +$$

$$+\frac{1}{2} \lim_{\delta \to +0} \int_{x_0-\delta}^{x_0+\delta} \left[\frac{\partial W(x,y_0+\delta)}{\partial y} - \frac{\partial W(x,y_0-\delta)}{\partial y}\right] dx = \alpha(h).$$

(2.4) $$\lim_{\delta \to +0} \int_{y_0-\delta}^{y_0+\delta} [W(x_0-\delta,y) - W(x_0+\delta,y)] dy = \beta(h),$$

(2.5) $$\lim_{\delta \to +0} \int_{x_0-\delta}^{x_0+\delta} [W(x,y_0-\delta) - W(x,y_0+\delta)] dx = \gamma(h).$$

Here $G(x,y,\xi,\eta)$ is Green function of the Dirichlet problem in Ω [9].

If we assume that $h(x,y) \in \mathcal{D}$ in a continuous manner in Ω_0 in the norm of $\mathbb{L}_2$ depends on the right side $f(x,y) \in \mathbb{L}_2(\Omega)$, then the outer boundary condition (2.2) takes the form

(2.6) $$W(x,y)|_{\partial\Omega} =< \Delta W(x,y), \sigma_0(x,y) >,$$

and the internal boundary conditions (2.3)-(2.5) will take the following types

(2.7) $$\frac{1}{2} \lim_{\delta \to +0} \int_{y_0-\delta}^{y_0+\delta} \left[\frac{\partial W(x_0+\delta,y)}{\partial x} - \frac{\partial W(x_0-\delta,y)}{\partial x}\right] dy +$$

$$+\frac{1}{2} \lim_{\delta \to +0} \int_{x_0-\delta}^{x_0+\delta} \left[\frac{\partial W(\xi,y_0+\delta)}{\partial y} - \frac{\partial W(\xi,y_0-\delta)}{\partial y}\right] dx =< \Delta W(x,y), \sigma_1(x,y) >.$$

(2.8) $$\lim_{\delta \to +0} \int_{y_0-\delta}^{y_0+\delta} [W(x_0-\delta,y) - W(x_0+\delta,y)] dy =< \Delta W(x,y), \sigma_2(x,y) >,$$

(2.8) $$\lim_{\delta \to +0} \int_{x_0-\delta}^{x_0+\delta} [W(x,y_0-\delta) - W(x,y_0+\delta)] dx =< \Delta W(x,y), \sigma_3(x,y) >,$$

where $\sigma_j(x,y) \in \mathbb{L}_2(\Omega), j = \overline{0,3}$.

For clarity of results, we assume that $\sigma_j(x,y) \equiv 0, j = 0,2,3$. For the operator L_{σ_1} the following theorem is true.

Theorem 1. Function

(2.10) $W(x,y) = \iint_\Omega G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta + < f(\xi,\eta), \sigma_1(\xi,\eta) > G(x,y,x_0,y_0)$

represents the only solution for all right-hand sides $f(x,y) \in \mathbb{L}_2(\Omega)$ of problem (1.1)-(1.5).
For prove theorem 1 we will use well known lemmas from [11]:

Lemma A. For any continuously differentiable function $g(x,y)$ the following equalities are true:

$$\lim_{\delta\to+0}\int_{y_0-\delta}^{y_0+\delta}\left[\frac{\partial g(x_0+\delta,y)}{\partial x}-\frac{\partial g(x_0-\delta,y)}{\partial x}\right]dy+$$

$$\lim_{\delta\to+0}\int_{x_0-\delta}^{x_0+\delta}\left[\frac{\partial g(x,y_0+\delta)}{\partial y}-\frac{\partial g(x,y_0-\delta)}{\partial y}\right]dx=0,$$

$$\lim_{\delta\to+0}\int_{y_0-\delta}^{y_0+\delta}[g(x_0-\delta,y)-g(x_0+\delta,y)]dy=0,$$

$$\lim_{\delta\to+0}\int_{x_0-\delta}^{x_0+\delta}[g(x,y_0-\delta)-g(x,y_0+\delta)]dx=0.$$

Lemma B. For function $G(x,y,x_0,y_0)$ the following equalities are true:

$$\alpha(G)=\frac{1}{2}\lim_{\delta\to+0}\int_{y_0-\delta}^{y_0+\delta}\left[\frac{\partial G(x_0+\delta,y,x_0,y_0)}{\partial x}-\frac{\partial G(x_0-\delta,y,x_0,y_0)}{\partial x}\right]dy+$$

$$+\frac{1}{2}\lim_{\delta\to+0}\int_{x_0-\delta}^{x_0+\delta}\left[\frac{\partial G(x,y_0+\delta,x_0,y_0)}{\partial y}-\frac{\partial G(x,y_0-\delta,x_0,y_0)}{\partial y}\right]dx=1,$$

$$\beta(G)=\lim_{\delta\to+0}\int_{x_0-\delta}^{x_0+\delta}\left[\frac{\partial G(x,y_0+\delta,x_0,y_0)}{\partial y}-\frac{\partial G(x,y_0-\delta,x_0,y_0)}{\partial y}\right]dx=0,$$

$$\gamma(G)=\lim_{\delta\to+0}\int_{x_0-\delta}^{x_0+\delta}[G(x,y_0-\delta,x_0,y_0)-G(x,y_0+\delta,x_0,y_0)]dx=0.$$

Now, we proof the theorem 1.

Proof. Let us show that equation (1.1) holds for $W(x,y)$. Operate the operator Laplace to the (2.10):

$$\Delta W(x,y)=\Delta\left(\iint_\Omega G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta\right)+$$

$$\langle f(\xi,\eta),\sigma_1(\xi,\eta)\rangle\Delta(G(x,y,x_0,y_0))=f(x,y)$$

when $(x,y)\in\Omega_0$, as by the Green function property $\Delta\left(\iint_\Omega G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta\right)=f(x,y)$ and $\Delta(G(x,y,x_0,y_0))=0$.

The validity of the first condition follows from the properties of the Green function.
Check the second condition

$$\alpha(W)-<\Delta W(x,y),\sigma_1(x,y)>=$$

$$\alpha\left(\iint_{\Omega} G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta\right) - < \Delta\left(\iint_{\Omega} G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta\right), \sigma_1(x,y) >$$

$$+ < f(\xi,\eta), \sigma(\xi,\eta) > \alpha\big(G(x,y,x_0,y_0)\big) - < f(\xi,\eta), \sigma_1(\xi,\eta) >$$

$$< \Delta G(x,y,x_0,y_0), \sigma_1(\xi,\eta) >= 0,$$

as by the Green function property $\Delta\left(\iint_{\Omega} G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta\right) = f(x,y)$ and $\Delta(G(x,y,x_0,y_0)) = 0$. And also by Lemma B $\alpha(G) = 1$. By Lemma B $\alpha\left(\iint_{\Omega} G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta\right) = 0$, as a function $u(x,y) = \alpha\left(\iint_{\Omega} G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta\right)$ is a twice differentiable function.

Check the third condition

$$\beta(W) = \beta\left(\iint_{\Omega} G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta\right) +$$

$$< f(\xi,\eta), \sigma_1(\xi,\eta) > \beta\big(G(x,y,x_0,y_0)\big) = 0.$$

By Lemma A $\beta\left(\iint_{\Omega} G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta\right) = 0$, as a function $u(x,y) = \alpha\left(\iint_{\Omega} G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta\right)$ is a twice differentiable function. And also by Lemma B $\beta(G(x,y,x_0,y_0)) = 0$.

Check the fourth condition

$$\gamma(W) = \gamma\left(\iint_{\Omega} G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta\right) +$$

$$< f(\xi,\eta), \sigma_1(\xi,\eta) > \gamma\big(G(x,y,x_0,y_0)\big) = 0.$$

By Lemma A $\gamma(G(x,y,x_0,y_0)) = 0$, as a function $u(x,y) = \alpha\left(\iint_{\Omega} G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta\right)$ is a twice differentiable function. By Lemma B $\gamma\left(\iint_{\Omega} G(x,y,\xi,\eta)f(\xi,\eta)d\xi d\eta\right) = 0$.

Theorem 1 is completely proved.

3 The resolvent of correct internal boundary value problems for the Laplace operator in a punctured domain

In this section in the functional space $\mathbb{W}_2^2(\Omega_0) \cap \mathbb{C}(\Omega)$, we calculate the explicit form of the resolvent for a wider class of operators $L_{\sigma_1\sigma_2\sigma_3} =: L$, generated by the differential equation (2.1) and internal boundary conditions (2.6)-(2.9) with $\sigma_0(\cdot,\cdot) \equiv 0$. The explicit form of the resolvent plays an essential role in studies of the spectral properties of the operator L. For convenience, we introduce the notation

$$T_1(x,y) := G(x,y,x_0,y_0), T_2(x,y) := \frac{\partial G(x,y,x_0,y_0)}{\partial \xi}, T_3(x,y) := \frac{\partial G(x,y,x_0,y_0)}{\partial \eta},$$

$$\kappa_j(x,y,\lambda) = L_0(L_0 - \lambda I)^{-1}T_j, j = 1,2,3.$$

We formulate the main result of this section.

Theorem 2. The resolvent of the operator L represents an operator-valued function of the spectral parameter λ and has the following representation:

$$(L - \lambda I)^{-1} f(x,y) = -\frac{H(f)}{d(\lambda)}, \quad (3.1)$$

$$H(f) = \begin{vmatrix} \kappa_1(x,y,\lambda) & \kappa_2(x,y,\lambda) & \kappa_3(x,y,\lambda) & (L_0-\lambda I)^{-1}f(x,y) \\ 1-\lambda\tilde{\alpha}(\kappa_1(x,y,\lambda)) & -\lambda\tilde{\alpha}(\kappa_2(x,y,\lambda)) & -\lambda\tilde{\alpha}(\kappa_3(x,y,\lambda)) & -\tilde{\alpha}(L_0(L_0-\lambda I)^{-1}f(x,y)) \\ -\lambda\tilde{\beta}(\kappa_1(x,y,\lambda)) & 1-\lambda\tilde{\beta}(\kappa_2(x,y,\lambda)) & -\lambda\tilde{\beta}(\kappa_3(x,y,\lambda)) & -\tilde{\beta}(L_0(L_0-\lambda I)^{-1}f(x,y)) \\ -\lambda\tilde{\gamma}(\kappa_1(x,y,\lambda)) & -\lambda\tilde{\gamma}(\kappa_2(x,y,\lambda)) & 1-\lambda\tilde{\gamma}(\kappa_3(x,y,\lambda)) & -\tilde{\gamma}(L_0(L_0-\lambda I)^{-1}f(x,y)) \end{vmatrix},$$

$$d(\lambda) = \begin{vmatrix} 1-\lambda\tilde{\alpha}(\kappa_1(x,y,\lambda)) & -\lambda\tilde{\alpha}(\kappa_2(x,y,\lambda)) & -\lambda\tilde{\alpha}(\kappa_3(x,y,\lambda)) \\ -\lambda\tilde{\beta}(\kappa_1(x,y,\lambda)) & 1-\lambda\tilde{\beta}(\kappa_2(x,y,\lambda)) & -\lambda\tilde{\beta}(\kappa_3(x,y,\lambda)) \\ -\lambda\tilde{\gamma}(\kappa_1(x,y,\lambda)) & -\lambda\tilde{\gamma}(\kappa_2(x,y,\lambda)) & 1-\lambda\tilde{\gamma}(\kappa_3(x,y,\lambda)) \end{vmatrix},$$

where

$$\kappa_j(x,y,\lambda) := L_0(L_0-\lambda I)^{-1}T_j(x,y),\ j=1,2,3,$$

$$\tilde{\alpha}\left(\kappa_j(x,y,\lambda)\right) = <\kappa_j(x,y,\lambda), \sigma_1(x,y)>, \tilde{\beta}\left(\kappa_j(x,y,\lambda)\right) =$$

$$<\kappa_j(x,y,\lambda), \sigma_2(x,y)>,$$

$$\tilde{\gamma}\left(\kappa_j(x,y,\lambda)\right) = <\kappa_j(x,y,\lambda), \sigma_3(x,y)>, \tilde{\alpha}(L_0(L_0-\lambda I)^{-1}f(x,y)) =$$

$$<L_0(L_0-\lambda I)^{-1}f(x,y), \sigma_1(x,y)>,$$

$$\tilde{\beta}(L_0(L_0-\lambda I)^{-1}f(x,y)) = <L_0(L_0-\lambda I)^{-1}f(x,y), \sigma_2(x,y)>,$$

$$\tilde{\gamma}(L_0(L_0-\lambda I)^{-1}f(x,y)) = <L_0(L_0-\lambda I)^{-1}f(x,y), \sigma_3(x,y)>.$$

Here L_0 is discred operator, corresponding to the Dirichlet problem.
Proof. In paper [12] the following relation was proved for the resolvent of the operator L:

(3.2) $$(L-\lambda I)^{-1}f(x,y) = (L_0-\lambda I)^{-1}f(x,y) +$$

$$\sum_{j=1}^{3} <L_0(L_0-\lambda I)^{-1}f(x,y), \sigma_j(x,y)> L(L-\lambda I)^{-1}T_j(x,y).$$

Set $f(x,y) = T_1(x,y)$. Then we have

$$(L-\lambda I)^{-1}T_1(x,y) = (L_0-\lambda I)^{-1}T_1(x,y) +$$

$$\sum_{j=1}^{3} <L_0(L_0-\lambda I)^{-1}T_1(x,y), \sigma_j(x,y)> L(L-\lambda I)^{-1}T_j(x,y).$$

We act on the obtained relation with the operator Δ. Recall that
1) when $u \in D(L)$ we have $\Delta u = Lu$;
2) when $u \in D(L_0)$ we obtain $\Delta u = L_0 u$. As a result, we get

(3.3) $$(1-\lambda <\kappa_1(x,y,\lambda), \sigma_1(x,y)>)L(L-\lambda I)^{-1}T_1(x,y) =$$

$$L_0(L_0-\lambda I)^{-1}T_1(x,y) + \sum_{j=2}^{3} \lambda <\kappa_1(x,y), \sigma_j(x,y)> L(L-\lambda I)^{-1}T_j(x,y).$$

Here it is taken into account that $(L_0-\lambda I)^{-1}T_1(x,y) \in D(L_0)$ and $(L-\lambda I)^{-1}T_j(x,y) \in D(L)$ when $j = 1,2,3$.

Now, assume $f(x,y)=T_2(x,y)$. Then from similar reasoning we have

$$(L-\lambda I)^{-1}T_2(x,y)=(L_0-\lambda I)^{-1}T_2(x,y)+$$

$$\sum_{j=1}^{3} < L_0(L_0-\lambda I)^{-1}T_2(x,y),\sigma_j(x,y) > L(L-\lambda I)^{-1}T_j(x,y).$$

We act on the obtained relation by the Laplace operator. As a result, we get that

(3.4) $$(1-\lambda < \kappa_2(x,y,\lambda),\sigma_2(x,y) >)L(L-\lambda I)^{-1}T_2(x,y)=$$

$$L_0(L_0-\lambda I)^{-1}T_2(x,y)+\lambda < \kappa_2(x,y),\sigma_1(x,y) > L(L-\lambda I)^{-1}T_1(x,y)$$

$$+\lambda < \kappa_2(x,y),\sigma_3(x,y) > L(L-\lambda I)^{-1}T_3(x,y).$$

When $f(x,y)=T_3(x,y)$, we have

$$(L-\lambda I)^{-1}T_3(x,y)=(L_0-\lambda I)^{-1}T_3(x,y)+$$

$$\sum_{j=1}^{3} < L_0(L_0-\lambda I)^{-1}T_3(x,y),\sigma_j(x,y) > L(L-\lambda I)^{-1}T_j(x,y).$$

We act on the obtained relation by the Laplace operator. As a result, we get that

(3.5) $$(1-\lambda < \kappa_3(x,y,\lambda),\sigma_1(x,y) >)L(L-\lambda I)^{-1}T_3(x,y)=$$

$$L_0(L_0-\lambda I)^{-1}T_3(x,y)++\sum_{j=1}^{2}\lambda < \kappa_3(x,y),\sigma_j(x,y) > L(L-\lambda I)^{-1}T_j(x,y).$$

Relations (3.2)-(3.5) constitute a homogeneous system of algebraic equations, which takes the matrix form

$$\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}=$$

$$=\begin{bmatrix}1 & -L(L-\lambda I)^{-1}T_1(x,y) & -L(L-\lambda I)^{-1}T_2(x,y) & -L(L-\lambda I)^{-1}T_3(x,y)\end{bmatrix}.$$

$$\cdot$$

$$\begin{bmatrix}\kappa_1(x,y,\lambda) & \kappa_2(x,y,\lambda) & \kappa_3(x,y,\lambda) & (L_0-\lambda I)^{-1}f(x,y)-(L-\lambda I)^{-1}f(x,y)\\ 1-\lambda\tilde{\alpha}(\kappa_1(x,y,\lambda)) & -\lambda\tilde{\alpha}(\kappa_2(x,y,\lambda)) & -\lambda\tilde{\alpha}(\kappa_3(x,y,\lambda)) & -\tilde{\alpha}(L_0(L_0-\lambda I)^{-1}f(x,y))\\ -\lambda\tilde{\beta}(\kappa_1(x,y,\lambda)) & 1-\lambda\tilde{\beta}(\kappa_2(x,y,\lambda)) & -\lambda\tilde{\beta}(\kappa_3(x,y,\lambda)) & -\tilde{\beta}(L_0(L_0-\lambda I)^{-1}f(x,y))\\ -\lambda\tilde{\gamma}(\kappa_1(x,y,\lambda)) & -\lambda\tilde{\gamma}(\kappa_2(x,y,\lambda)) & 1-\lambda\tilde{\gamma}(\kappa_3(x,y,\lambda)) & -\tilde{\gamma}(L_0(L_0-\lambda I)^{-1}f(x,y))\end{bmatrix}$$

It is known from the course of linear algebra that a homogeneous system of equations has a non-trivial solution for

$$\begin{vmatrix} \kappa_1(x,y,\lambda) & \kappa_2(x,y,\lambda) & \kappa_3(x,y,\lambda) & (L_0-\lambda I)^{-1}f(x,y)-(L-\lambda I)^{-1}f(x,y) \\ 1-\lambda\tilde{\alpha}(\kappa_1(x,y,\lambda)) & -\lambda\tilde{\alpha}(\kappa_2(x,y,\lambda)) & -\lambda\tilde{\alpha}(\kappa_3(x,y,\lambda)) & -\tilde{\alpha}(L_0(L_0-\lambda I)^{-1}f(x,y)) \\ -\lambda\tilde{\beta}(\kappa_1(x,y,\lambda)) & 1-\lambda\tilde{\beta}(\kappa_2(x,y,\lambda)) & -\lambda\tilde{\beta}(\kappa_3(x,y,\lambda)) & -\tilde{\beta}(L_0(L_0-\lambda I)^{-1}f(x,y)) \\ -\lambda\tilde{\gamma}(\kappa_1(x,y,\lambda)) & -\lambda\tilde{\gamma}(\kappa_2(x,y,\lambda)) & 1-\lambda\tilde{\gamma}(\kappa_3(x,y,\lambda)) & -\tilde{\gamma}(L_0(L_0-\lambda I)^{-1}f(x,y)) \end{vmatrix} = 0.$$

According to the standard properties of the determinants, we can write the equality

$$\begin{vmatrix} \kappa_1(x,y,\lambda) & \kappa_2(x,y,\lambda) & \kappa_3(x,y,\lambda) & (L-\lambda I)^{-1}f(x,y) \\ 1-\lambda\tilde{\alpha}(\kappa_1(x,y,\lambda)) & -\lambda\tilde{\alpha}(\kappa_2(x,y,\lambda)) & -\lambda\tilde{\alpha}(\kappa_3(x,y,\lambda)) & 0 \\ -\lambda\tilde{\beta}(\kappa_1(x,y,\lambda)) & 1-\lambda\tilde{\beta}(\kappa_2(x,y,\lambda)) & -\lambda\tilde{\beta}(\kappa_3(x,y,\lambda)) & 0 \\ -\lambda\tilde{\gamma}(\kappa_1(x,y,\lambda)) & -\lambda\tilde{\gamma}(\kappa_2(x,y,\lambda)) & 1-\lambda\tilde{\gamma}(\kappa_3(x,y,\lambda)) & 0 \end{vmatrix} =$$

$$= \begin{vmatrix} \kappa_1(x,y,\lambda) & \kappa_2(x,y,\lambda) & \kappa_3(x,y,\lambda) & (L_0-\lambda I)^{-1}f(x,y) \\ 1-\lambda\tilde{\alpha}(\kappa_1(x,y,\lambda)) & -\lambda\tilde{\alpha}(\kappa_2(x,y,\lambda)) & -\lambda\tilde{\alpha}(\kappa_3(x,y,\lambda)) & -\tilde{\alpha}(L_0(L_0-\lambda I)^{-1}f(x,y)) \\ -\lambda\tilde{\beta}(\kappa_1(x,y,\lambda)) & 1-\lambda\tilde{\beta}(\kappa_2(x,y,\lambda)) & -\lambda\tilde{\beta}(\kappa_3(x,y,\lambda)) & -\tilde{\beta}(L_0(L_0-\lambda I)^{-1}f(x,y)) \\ -\lambda\tilde{\gamma}(\kappa_1(x,y,\lambda)) & -\lambda\tilde{\gamma}(\kappa_2(x,y,\lambda)) & 1-\lambda\tilde{\gamma}(\kappa_3(x,y,\lambda)) & -\tilde{\gamma}(L_0(L_0-\lambda I)^{-1}f(x,y)) \end{vmatrix}$$

This implies the assertion of Theorem 2.

From the explicit resolvent formula for the operator L, it is not difficult to see that the resolvent is a meromorphic operator-valued function, since the characteristic determinant $d(\lambda)$ may have poles in the spectrum of the operator L_0. Since the spectrum of the Dirichlet problem for the Laplace equation is canonical restricted domains can be explicitly computed, then these poles are written out explicitly.

Corollary 1. The resolvent of the operator L can also be represented as

(3.6) $$(L-\lambda I)^{-1}f(x,y) = (L_0-\lambda I)^{-1}f(x,y) - \frac{\tilde{H}(f)}{d(\lambda)}$$

where

$$\tilde{H}(f) = \begin{bmatrix} \kappa_1(x,y,\lambda) & \kappa_2(x,y,\lambda) & \kappa_3(x,y,\lambda) & 0 \\ 1-\lambda\tilde{\alpha}(\kappa_1(x,y,\lambda)) & -\lambda\tilde{\alpha}(\kappa_2(x,y,\lambda)) & -\lambda\tilde{\alpha}(\kappa_3(x,y,\lambda)) & -\tilde{\alpha}(L_0(L_0-\lambda I)^{-1}f(x,y)) \\ -\lambda\tilde{\beta}(\kappa_1(x,y,\lambda)) & 1-\lambda\tilde{\beta}(\kappa_2(x,y,\lambda)) & -\lambda\tilde{\beta}(\kappa_3(x,y,\lambda)) & -\tilde{\beta}(L_0(L_0-\lambda I)^{-1}f(x,y)) \\ -\lambda\tilde{\gamma}(\kappa_1(x,y,\lambda)) & -\lambda\tilde{\gamma}(\kappa_2(x,y,\lambda)) & 1-\lambda\tilde{\gamma}(\kappa_3(x,y,\lambda)) & -\tilde{\gamma}(L_0(L_0-\lambda I)^{-1}f(x,y)) \end{bmatrix}$$

Corollary 2. In particular, if $\sigma_j(x,y) \equiv 0, j = 2,3$, then the operator's resolution L_{σ_1} take the form

(3.7) $$(L_{\sigma_1}-\lambda I)^{-1}f(x,y) = (L_0-\lambda I)^{-1}f(x,y) + \frac{\tilde{\alpha}(\kappa_1(x,y,\lambda))}{d(\lambda)}\kappa_1(x,y,\lambda),$$

where

(3.8) $$d(\lambda) = 1-\lambda\tilde{\alpha}(\kappa_1(x,y,\lambda)) = 1-\lambda < \kappa_1(x,y,\lambda), \sigma(x,y) >.$$

4 Meromorphic function generating root functions operator L_{σ_1}

In further research we will need some properties $\kappa_1(x,y,\lambda)$.

Lemma 1. Meromorphic function $\kappa_1(x,y,\lambda)$ is the main solution of the homogeneous equation

$$\Delta(\kappa_1(x,y,\lambda)) = \lambda\kappa_1(x,y,\lambda),$$

satisfying linear conditions

$$\kappa_1(x,y,\lambda)|_{\partial\Omega} = 0,$$

$$\alpha(\kappa_1(x,y,\lambda)) - < \Delta(\kappa_1(x,y,\lambda)), \sigma_1(x,y) >= d(\lambda),$$

$$\beta(\kappa_1(x,y,\lambda)) = 0,$$

$$\gamma(\kappa_1(x,y,\lambda)) = 0,$$

where $d(\lambda)$ is define by (3.8).

Proof. Check that the function $\kappa_1(x,y,\lambda)$ is a solution of the homogeneous equation

$$\Delta(\kappa_1(x,y,\lambda)) = \Delta(T_1(x,y)) + \lambda L_0(L_0 - \lambda I)^{-1}T_1(x,y) = \lambda\kappa_1(x,y,\lambda).$$

Here it is taken into account that $\Delta(T_1(x,y)) = 0$ и $(L_0 - \lambda I)^{-1}T_1(x,y) \in D(L_0)$.
The validity of the first condition follows from the properties of the Green function.
Let us prove the validity of the second condition.

$$\alpha(\kappa_1(x,y,\lambda)) - < \Delta(\kappa_1(x,y,\lambda)), \sigma_1(x,y) >= \alpha(T_1(x,y)) +$$

$$\lambda\alpha((L_0 - \lambda I)^{-1}T_1(x,y)) - < \Delta(T_1(x,y)), \sigma_1(x,y) > -$$

$$\lambda < L_0(L_0 - \lambda I)^{-1}T_1(x,y), \sigma_1(x,y) >= d(\lambda),$$

so by Lemma B $\alpha(T_1(x,y)) = 1$. From Lemma A it follows that $\alpha((L_0 - \lambda I)^{-1}T_1(x,y)) = 0$. It is also taken into account that $\Delta(T_1(x,y)) = 0$ and $(L_0 - \lambda I)^{-1}T_1(x,y) \in D(L_0)$.
Check the third condition

$$\beta(\kappa_1(x,y,\lambda)) = \beta(T_1(x,y)) + \lambda\beta((L_0 - \lambda I)^{-1}T_1(x,y)) = 0,$$

as by Lemma B $\beta(T_1(x,y)) = 0$. From Lemma A it follows that $\beta((L_0 - \lambda I)^{-1}T_1(x,y)) = 0$.
Check the fourth condition

$$\gamma(\kappa_1(x,y,\lambda)) = \gamma(T_1(x,y)) + \lambda\gamma((L_0 - \lambda I)^{-1}T_1(x,y)) = 0,$$

as by Lemma B $\gamma(T_1(x,y)) = 0$. From Lemma A it follows that $\gamma((L_0 - \lambda I)^{-1}T_1(x,y)) = 0$.
Lemma 1 is proved.
For convenience, we introduce the notation

(4.1) $$\varphi_1(x,y,\lambda) = \frac{\kappa_1(x,y,\lambda)}{d(\lambda)}.$$

If relation (4.1) is taken into account, then formula (3.7) for the resolvent of the operator L_{σ_1} can be written in the following form

(4.2) $$(L_{\sigma_1} - \lambda I)^{-1}f(x,y) = (L_0 - \lambda I)^{-1}f(x,y) + \tilde{\alpha}(\kappa_1(x,y,\lambda))\varphi(x,y,\lambda).$$

This implies that the poles of $\varphi_1(x,y,\lambda)$, and at the same time, the poles of the resolvent $(L_{\sigma_1} - \lambda I)^{-1}$ coincide with the zeros of the meromorphic function $d(\lambda)$. We show that the introduced solution $\kappa_1(x,y,\lambda)$ as a function of λ on the spectrum of the operator L_{σ_1} generates all its root functions.

Theorem 3. Let λ_k be an arbitrary zero of the characteristic determinant $d(\lambda)$ of multiplicity m_k. Then the functions from the following line:

(4.3) $$\kappa_1(x,y,\lambda_k),\frac{1}{1!}\frac{\partial\kappa_1(x,y,\lambda_k)}{\partial\lambda},\ldots,\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}\kappa_1(x,y,\lambda_k)}{\partial\lambda^{m_k-1}},$$

have a spectral interpretation: the first nonzero zero function is a proper one, and the subsequent ones are attached, generated by the indicated eigenfunction, corresponding to the eigenvalue λ_k of the operator L_{σ_1}.

Proof. The proof of Theorem 3 is that it is necessary to check for $p=0,\ldots,m_k-1$ relations

(4.4) $$\frac{1}{p!}\frac{\partial^p\kappa_1(x,y,\lambda_k)}{\partial\lambda^p}\in D(L_{\sigma_1}),$$

(4.5) $$L_{\sigma_1}\left(\frac{1}{p!}\frac{\partial^p\kappa_1(x,y,\lambda_k)}{\partial\lambda^p}\right)=\lambda_k\left(\frac{1}{p!}\frac{\partial^p\kappa_1(x,y,\lambda_k)}{\partial\lambda^p}\right)+\frac{\epsilon}{(p-1)!}\frac{\partial^{p-1}\kappa_1(x,y,\lambda_k)}{\partial\lambda^{p-1}},$$

where $\epsilon=0$ when $p=0$, $\epsilon=1$ when $p>0$. From what $d(\lambda_k)=0$ and follows from Lemma 1 $\kappa_1(x,y,\lambda_k)\in D(L_{\sigma_1})$. Therefore, the operator relation $L_{\sigma_1}(\kappa_1(x,y,\lambda_k))=\lambda_k\kappa_1(x,y,\lambda_k)$ coincides with a homogeneous differential equation $\Delta(\kappa_1(x,y,\lambda_k))=\lambda_k\kappa_1(x,y,\lambda_k)$, which by definition $\kappa(x,y,\lambda)$ holds when $\lambda=\lambda_k$. Thus, if $\kappa(x,y,\lambda_k)$ is not identically zero functions, then $\kappa(x,y,\lambda_k)$ is eigenfunction of the operator L_{σ_1}.

Now let $p=1<m_k$. Notice, that $d(\lambda_k)=0$, $d'(\lambda_k)=0$ and

$$\frac{\partial\kappa_1(x,y,\lambda)}{\partial\lambda}=(L_0-\lambda I)^{-1}T_1(x,y)+\lambda\frac{\partial}{\partial\lambda}(L_0-\lambda I)^{-1}T_1(x,y),$$

$$d'(\lambda)=-<\kappa_1(x,y,\lambda),\sigma_1(x,y)>-\lambda<\frac{\partial}{\partial\lambda}\kappa_1(x,y,\lambda),\sigma(x,y)>.$$

Calculate

$$\Delta\left(\frac{\partial\kappa_1(x,y,\lambda)}{\partial\lambda}\right)=L_0(L_0-\lambda I)^{-1}T_1(x,y)+\lambda\frac{\partial}{\partial\lambda}L_0(L_0-\lambda I)^{-1}T_1(x,y)=$$

$$=\lambda\frac{\partial}{\partial\lambda}\kappa_1(x,y,\lambda)+\kappa_1(x,y,\lambda).$$

The validity of the first condition follows from the properties of the Green function.

Let us prove the validity of the second condition.

$$\alpha\left(\frac{\partial\kappa_1(x,y,\lambda)}{\partial\lambda}\right)-<\Delta\left(\frac{\partial\kappa_1(x,y,\lambda)}{\partial\lambda}\right),\sigma_1(x,y)>=\alpha((L_0-\lambda_k)^{-1}T_1(x,y))+$$

$$\lambda_k\frac{\partial}{\partial\lambda}\alpha((L_0-\lambda_k)^{-1}T_1(x,y))-<\Delta((L_0-\lambda_k)^{-1}T_1(x,y)),\sigma_1(x,y)>$$

$$-\lambda_k\frac{\partial}{\partial\lambda}<\Delta((L_0-\lambda_k)^{-1}T_1(x,y)),\sigma_1(x,y)>=d'(\lambda_k)=0,$$

so by Lemma A $\alpha((L_0-\lambda_k I)^{-1}T_1(x,y))=0$. It is also taken into account that $(L_0-\lambda_k I)^{-1}T_1(x,y)\in D(L_0)$.

Check the third condition

$$\beta\left(\frac{\partial\kappa_1(x,y,\lambda)}{\partial\lambda}\right)=\beta((L_0-\lambda_k)^{-1}T_1(x,y))+\lambda_k\frac{\partial}{\partial\lambda}\beta((L_0-\lambda_k)^{-1}T_1(x,y))=0,$$

so by Lemma A $\beta((L_0-\lambda I)^{-1}T_1(x,y))=0$.

Check the fourth condition

$$\gamma(\kappa_1(x,y,\lambda))=\gamma((L_0-\lambda_k)^{-1}T_1(x,y))+\lambda_k\frac{\partial}{\partial\lambda}\gamma((L_0-\lambda_k)^{-1}T_1(x,y))=0,$$

so by Lemma A $\gamma((L_0-\lambda I)^{-1}T_1(x,y))=0$.

Continuing the arguments for other admissible p, we obtain a complete proof of Theorem 3.

5 Projectors on rood supspace

In the monograph ([13], стр. 445) the decomposition theorem is given, from which it follows that the projector $P_k: \mathbb{L}_2(\Omega_0) \to (L_{\sigma_1} - \lambda_k I)^{m_k}$ represents the residue of the resolvent at the singular point λ_k:

$$(P_k f)(x,y) = -\frac{1}{2\pi i} \oint\limits_{|\lambda-\lambda_k|=\delta} (L_{\sigma_1} - \lambda I)^{-1} f(x,y)\, d\lambda$$

with some $\delta > 0$. Since the resolvent of the Dirichlet operator has poles, for completeness of information we consider two cases.

First, we consider the case when the eigenvalue λ_k of the operator L_{σ_1} does not coincide with any eigenvalue λ_j^0 of the operator L_0. Recalling the representation of the resolvent (3.7) from Corollary 2 and considering that the resolvent $(L_0 - \lambda I)^{-1}$ of the Dirichlet operator represents a holomorphic function of λ, projector type P_k can be specified:

(5.1) $$(P_k f)(x,y) = -\frac{1}{2\pi i} \oint_{|\lambda-\lambda_k|=\delta} \frac{\kappa_1(x,y,\lambda)}{d(\lambda)}$$

$$< f(\xi,\eta), L_0^*(L_0^* - \overline{\lambda} I)^{-1} \sigma_1(\xi,\eta) > d\lambda = \underset{\lambda_k}{res} \frac{\kappa_1(x,y,\lambda)}{d(\lambda)} < f(\xi,\eta), M_1(\xi,\eta,\overline{\lambda}) >,$$

where $M_1(\xi,\eta,\overline{\lambda}) = \overline{L_0^*(L_0^* - \overline{\lambda} I)^{-1} \sigma_1(\xi,\eta)}$.

Apply the Cauchy residue theorem to the relation (5.1), we obtain

(5.2) $$(P_k f)(x,y) = -\frac{1}{(m_k-1)!} \lim_{\lambda\to\lambda_k} \frac{\partial^{m_k-1}}{\partial\lambda^{m_k-1}} \left(\frac{(\lambda-\lambda_k)^{m_k} < f(\xi,\eta), M_1(\xi,\eta,\overline{\lambda}) >}{d(\lambda)} \kappa_1(x,y,\lambda) \right) =$$

$$-\sum_{p=0}^{m_k-1} \frac{1}{(m_k-1-p)!} \lim_{\lambda\to\lambda_k} \frac{\partial^{m_k-1-p}}{\partial\lambda^{m_k-1-p}} \left(\frac{(\lambda-\lambda_k)^{m_k} < f(\xi,\eta), M_1(\xi,\eta,\overline{\lambda}) >}{d(\lambda)} \right) \frac{1}{p!} \frac{\partial^p \kappa_1(x,y,\lambda)}{\partial\lambda^p}\Big|_{\lambda=\lambda_k} =$$

$$\sum_{p=0}^{m_k-1} < f(\xi,\eta), -\frac{1}{(m_k-1-p)!} \lim_{\overline{\lambda}\to\overline{\lambda}_k} \frac{\partial^{m_k-1-p}}{\partial\overline{\lambda}^{m_k-1-p}} \left(\frac{(\overline{\lambda}-\overline{\lambda}_k)^{m_k} M_1(\xi,\eta,\overline{\lambda})}{\overline{d(\lambda)}} \right) >$$

$$\frac{1}{p!} \frac{\partial^p \kappa_1(x,y,\lambda)}{\partial\lambda^p}\Big|_{\lambda=\lambda_k}$$

Analysis of formula (5.2) leads to the following notation.

(5.3) $$E_k' = \{h_{k,0}(\xi,\eta), h_{k,1}(\xi,\eta), \dots, h_{k,m_k-1}(\xi,\eta)\},$$

where

(5.4) $$h_{k,m_k-1-p}(\xi,\eta) = -\frac{1}{(m_k-1-p)!} \lim_{\overline{\lambda}\to\overline{\lambda}_k} \frac{\partial^{m_k-1-p}}{\partial\overline{\lambda}^{m_k-1-p}} \left(\frac{(\overline{\lambda}-\overline{\lambda}_k)^{m_k} M_1(\xi,\eta,\overline{\lambda})}{\overline{d(\lambda)}} \right),$$

$$p = \overline{0, m_k - 1}.$$

We introduce the following family of functions

(5.5) $$E' = \{E_k' : \lambda_k \text{ is arbitrary eigenvalue of the operator } L_{\sigma_1}\}.$$

Thus, we need to study the decomposition of arbitrary elements from the functional space $\mathbb{L}_2(\Omega_0)$ in the system of root functions of the operator L_{σ_1}.

(5.6) $$E = \{E_k \colon \lambda_k \text{ is arbitrary eigenvalue of the operator } \mathrm{L}_{\sigma_1}\}.$$

(5.7) $$E_k = \{W_{k,0}(x,y), W_{k,1}(x,y), \dots, W_{k,m_k-1}(x,y)\},$$

where

$$W_{k,p}(x,y) = \left.\frac{\partial^p \kappa_1(x,y,\lambda)}{\partial \lambda^p}\right|_{\lambda=\lambda_k}$$

Now, we consider the case $\lambda_k = \lambda_k^0$.

$$(P_k f)(x,y) = -\frac{1}{2\pi i}\oint_{|\lambda-\lambda_k|=\delta} (L_{\sigma_1} - \lambda I)^{-1} f(x,y)\, d\lambda =$$

$$= \underset{\lambda_k=\lambda_k^0}{res}((L_0 - \lambda I)^{-1} f(x,y)) +$$

$$\underset{\lambda_k=\lambda_k^0}{res}\left(\tilde{\alpha}(L_0(L_0 - \lambda I)^{-1} f(x,y))\frac{\kappa_1(x,y,\lambda)}{1 - \lambda\tilde{\alpha}(\kappa_1(x,y,\lambda))}\right).$$

First, we calculate the first addend. Given the representation for the resolvent of the Dirichlet problem as an expansion in eigenfunctions

(5.8) $$(L_0 - \lambda I)^{-1} f(x,y) = \sum_{m=1}^{\infty} \frac{1}{\lambda_m^0 - \lambda} f_m \omega_m^0(x,y),$$

we calculate the first addend

$$V_1 := \underset{\lambda_k=\lambda_k^0}{res}((L_0 - \lambda I)^{-1} f(x,y)) =$$

$$\underset{\lambda_k=\lambda_k^0}{res}\left(\sum_{m=1}^{\infty} \frac{1}{\lambda_m^0 - \lambda} f_m \omega_m^0(x,y)\right) = -f_k \omega_k^0(x,y).$$

Calculate the second addend. Here we use the standard transform.

$$L_0(L_0 - \lambda I)^{-1} T_1(x,y) = T_1(x,y) + \lambda(L_0 - \lambda I)^{-1} T_1(x,y).$$

Then

$$V_2 := \underset{\lambda_k=\lambda_k^0}{res}\left(\left[\tilde{\alpha}(f) + \sum_{m=1}^{\infty} \frac{\lambda}{\lambda_m^0 - \lambda} f_m \tilde{\alpha}(\omega_m^0(x,y))\right]\right.$$

$$\left.\frac{T_1(x,y) + \sum_{m=1}^{\infty} \frac{\lambda}{\lambda_m^0 - \lambda} T_{1m}\omega_m^0(x,y)}{1 - \lambda\tilde{\alpha}(T_1(x,y)) - \sum_{m=1}^{\infty} \frac{\lambda^2}{\lambda_m^0} T_{1m}\tilde{\alpha}(\omega_m^0(x,y))}\right).$$

Multiply the denominator and numerator in the last relation by $\lambda_k - \lambda$. As a result, the deduction will be equal to

$$V_2 = \lambda_k^0 f_k \tilde{\alpha}(\omega_k^0(x,y)) \cdot \frac{\lambda_k^0 T_{1k} \omega_k^0(x,y)}{(\lambda_k^0)^2 T_{1k} \tilde{\alpha}(\omega_k^0(x,y))} = f_k \omega_k^0(x,y).$$

So in the case of $\lambda_k = \lambda_k^0$ the projector

$$(P_k f)(x,y) = V_1 + V_2 = -f_k \omega_k^0(x,y) + f_k \omega_k^0(x,y) = 0.$$

The main result of this section is formulated as a theorem.

Theorem 4. Let λ_k is eigenvalue of the multiplicity m_k of the operator L_{σ_1}. P_k operator on root the subspace of the operator L_{σ_1} corresponding to λ_k is determined by formula (5.2).

6 Minimality of the root function system

In this section we prove the minimality of the system of root functions of the operator L_{σ_1} in the functional space $\mathbb{L}_2(\Omega_0)$. Generally speaking, choosing one or another basis in the root subspaces $P_k\mathbb{L}_2(\Omega_0)$, it is possible to study different systems of root functions. We investigate the minimality of a concrete system (5.6). System (5.6) is generated by solutions of the differential equation (4.5), generating operator L_{σ_1}. In particular, the Green's function associated with the operator L_{σ_1} is written through them. To prove the minimality of (5.6) in the functional space $\mathbb{L}_2(\Omega_0)$, it suffices to construct a biorthogonal system of functions in this space. In the previous paragraph 5 such a system has already been built. It remains to verify that system (5.6) lies in $\mathbb{L}_2(\Omega_0)$ and satisfies the biorthogonality relation.

To begin with, we verify that the functions in (5.6) are elements of the space $\mathbb{L}_2(\Omega_0)$. Since, according to (5.4), each $h_{k,t}(\xi,\eta)$ represents a linear combination of functions $M_1(\xi,\eta,\overline{\lambda})$ and its derivatives for $\lambda = \lambda_k$ $(k = 1,2,\dots,n)$, it suffices to check that these functions $\overline{M_1(\xi,\eta,\overline{\lambda_k}),\ (k =}$ $1,2,\dots,n)$ belong to $\mathbb{L}_2(\Omega_0)$. The latter fact is obvious, since $M_1(\xi,\eta,\overline{\lambda}) = L_0^*(L_0^* - \overline{\lambda} I)^{-1}\sigma_1(\xi,\eta)$ and $\sigma_1(\xi,\eta)$ are an element of $\mathbb{L}_2(\Omega)$.

To establish the biorthogonality relations, we need the following lemma.

Lemma 2. For arbitrary complex numbers λ,μ the identity is correct

$$< \kappa_1(x,y,\lambda), M_1(x,y,\overline{\mu}) >= -\frac{d(\lambda)-d(\mu)}{\lambda-\mu}.$$

Proof. For arbitrary λ,μ we calculate the following inner product

$$\lambda < \kappa_1(x,y,\lambda), M_1(x,y,\overline{\mu}) >_{\mathbb{L}_2(\Omega_0)} = \lambda < \kappa_1(x,y,\lambda), \sigma_1(x,y) +$$

$$\mu(L_0^* - \overline{\mu} I)^{-1}\sigma_1(x,y) \geq < \Delta\kappa_1(x,y,\lambda), \sigma_1(x,y) > +$$

$$\mu < \Delta\kappa_1(x,y,\lambda), (L_0^* - \overline{\mu} I)^{-1}\sigma_1(x,y) >$$

$$= 1 - d(\lambda) + \mu < \Delta\kappa_1(x,y,\lambda), \widetilde{M_1}(x,y,\overline{\mu}) >,$$

where $\widetilde{M_1}(x,y,\overline{\mu}) = (L_0^* - \overline{\mu} I)^{-1}\sigma_1(x,y)$. Here, took into account the formula (3.8) for the characteristic determinant operator L_{σ_1}. We calculate separately the scalar product

$$< \Delta\kappa_1(x,y,\lambda), \widetilde{M_1}(x,y,\overline{\mu}) >=< \kappa_1(x,y,\lambda), \Delta\widetilde{M_1}(x,y,\overline{\mu}) > +$$

$$+ \lim_{\delta\to+0} \int_{\partial\Omega} \left(\frac{\partial\kappa_1(x,y,\lambda)}{\partial\overline{n}} \widetilde{M_1}(x,y,\overline{\mu}) - \kappa_1(x,y,\lambda) \frac{\partial\widetilde{M_1}(x,y,\overline{\mu})}{\partial\overline{n}} \right) ds -$$

$$-\lim_{\delta\to+0}\int_{\partial\prod_\delta^0}\left(\frac{\partial\kappa_1(x,y,\lambda)}{\partial\overline{n}}\widetilde{M_1}(x,y,\overline{\mu})-\kappa_1(x,y,\lambda)\frac{\partial\widetilde{M_1}(x,y,\overline{\mu})}{\partial\overline{n}}\right)ds=$$

$$=<\kappa_1(x,y,\lambda),\Delta\widetilde{M_1}(x,y,\overline{\mu})>-$$

$$\lim_{\delta\to+0}\int_{\partial\prod_\delta^0}\left(\frac{\partial\kappa_1(x,y,\lambda)}{\partial\overline{n}}-\kappa_1(x,y,\lambda)\frac{\partial\widetilde{M_1}(x,y,\overline{\mu})}{\partial\overline{n}}\right)ds,$$

in that $\kappa_1(x,y,\lambda),\widetilde{M_1}(x,y,\overline{\mu})\in\mathbb{W}_2^2(\Omega)$, then

$$\lim_{\delta\to+0}\int_{\partial\Omega}\left(\frac{\partial\kappa_1(x,y,\lambda)}{\partial\overline{n}}\widetilde{M_1}(x,y,\overline{\mu})-\kappa_1(x,y,\lambda)\frac{\partial\widetilde{M_1}(x,y,\overline{\mu})}{\partial\overline{n}}\right)ds=0.$$

Here $\prod_\delta^0=\{(x,y):-\delta\le x-x_0\le\delta,-\delta\le y-y_0\le\delta\}$. Then

$$(\lambda-\mu)<\kappa_1(x,y,\lambda),M_1(x,y,\overline{\mu})>=1-d(\lambda)-$$

$$-\mu\lim_{\delta\to+0}\int_{\partial\prod_\delta^0}\left(\frac{\partial\kappa_1(x,y,\lambda)}{\partial\overline{n}}\widetilde{M_1}(x,y,\overline{\mu})-\kappa_1(x,y,\lambda)\frac{\partial\widetilde{M_1}(x,y,\overline{\mu})}{\partial\overline{n}}\right)ds.$$

Notice, that $\Delta\widetilde{M_1}(x,y,\overline{\mu})=M_1(x,y,\overline{\mu})$, because $(L_0^*-\overline{\mu}I)^{-1}\sigma_1(x,y)\in D(L_0^*)$. Hence, when $\lambda=\mu$, we have

$$\mu\lim_{\delta\to+0}\int_{\partial\prod_\delta^0}\left(\frac{\partial\kappa_1(x,y,\mu)}{\partial\overline{n}}\widetilde{M_1}(x,y,\overline{\mu})-\kappa_1(x,y,\mu)\frac{\partial\widetilde{M_1}(x,y,\overline{\mu})}{\partial\overline{n}}\right)ds=1-d(\mu).$$

Considering the last relation, we transform the previous equality

$$(\lambda-\mu)<\kappa_1(x,y,\lambda),M_1(x,y,\overline{\mu})>=1-d(\lambda)-1+d(\mu)-$$

$$\mu\lim_{\delta\to+0}\int_{\partial\prod_\delta^0}\left[\left(\frac{\partial\kappa_1(x,y,\lambda)}{\partial\overline{n}}-\frac{\partial\kappa_1(x,y,\mu)}{\partial\overline{n}}\right)\widetilde{M_1}(x,y,\overline{\mu})-\right.$$

$$\left.(\kappa_1(x,y,\lambda)-\kappa_1(x,y,\mu))\frac{\partial\widetilde{M_1}(x,y,\overline{\mu})}{\partial\overline{n}}\right]ds=-d(\lambda)+d(\mu)-$$

$$-(\lambda-\mu)\mu\lim_{\delta\to+0}\int_{\partial\prod_\delta^0}\frac{\partial}{\partial\overline{n}}(L_0-\lambda I)^{-1}T_1(x,y)\cdot(L_0^*-\overline{\mu}I)^{-1}\sigma_1(x,y)ds+$$

$$+(\lambda-\mu)\mu\lim_{\delta\to+0}\int_{\partial\prod_\delta^0}(L_0-\lambda I)^{-1}T_1(x,y)\cdot\frac{\partial}{\partial\overline{n}}(L_0^*-\overline{\mu}I)^{-1}\sigma_1(x,y)ds$$

Recall the representation for the Dirichlet problem as an expansion in eigenfunctions

$$(L_0-\lambda I)^{-1}T_1(x,y)=\sum_{m=1}^{\infty}\frac{<T_1(x,y),\omega_m^0(x,y)>}{\lambda_m^0-\lambda}\omega_m^0(x,y)=$$

$$\sum_{m=1}^{\infty}\frac{T_{1m}}{\lambda_m^0-\lambda}\omega_m^0(x,y),$$

$$(L_0^*-\overline{\mu}I)^{-1}\sigma_1(x,y)=\sum_{n=1}^{\infty}\frac{<\sigma_1(x,y),\omega_n^0(x,y)>}{\lambda_n^0-\overline{\mu}}\omega_n^0(x,y)=\sum_{n=1}^{\infty}\frac{\sigma_{1n}}{\lambda_n^0-\overline{\mu}}\omega_n^0(x,y).$$

As a result, we have

$$(\lambda-\mu) < \kappa_1(x,y,\lambda), M_1(x,y,\overline{\mu}) >= -d(\lambda)+d(\mu)-$$

$$-(\lambda-\mu)\mu\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{T_{1m}\sigma_{1n}}{(\lambda_m^0-\lambda)(\lambda_n^0-\mu)}$$

$$\lim_{\delta\to+0}\int\limits_{\partial\Pi_\delta^0}\left[\frac{\partial}{\partial\overline{n}}\omega_m^0\omega_n^0(x,y)-\omega_m^0(x,y)\frac{\partial}{\partial\overline{n}}\omega_n^0(x,y)\right]ds=-d(\lambda)+d(\mu)-$$

$$(\lambda-\mu)\mu\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{T_{1m}\sigma_{1n}}{(\lambda_m^0-\lambda)(\lambda_n^0-\mu)}$$

$$\lim_{\delta\to+0}\int\limits_{\partial\Pi_\delta^0}[\Delta\omega_m^0\omega_n^0(x,y)-\omega_m^0(x,y)\Delta\omega_n^0(x,y)]ds=-d(\lambda)+d(\mu)-$$

$$(\lambda-\mu)\mu\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{T_{1m}\sigma_{1n}}{(\lambda_m^0-\lambda)(\lambda_n^0-\mu)}(\lambda_m^0-\lambda_n^0)$$

$$\lim_{\delta\to+0}\int\limits_{\Pi_\delta^0}\omega_m^0\omega_n^0(x,y)ds=-d(\lambda)+d(\mu)$$

as if $m=n$, то $\lambda_m^0-\lambda_n^0=0$. If $m\neq n$, then the integral is equal to zero due to the orthogonality of the eigenfunctions of the operator L_0.

Lemma 2 is proved.

To establish the biorthogonality relations, we check the

$$< W_{n,s}(x,y), h_{k,m_k-1-p}(x,y) >= \begin{cases}1, & \text{если } (n,s)=(k,p);\\ 0, & \text{если } (n,s)\neq(k,p).\end{cases}$$

We consider two eigenvalue λ_s and λ_k. They match pairs (s,t) and (k,p), where $t=0,1,\dots,m_s-1$ and $p=0,1,\dots,m_k-1$. Note that the inner product

$$< W_{n,s}(x,y), h_{k,m_k-1-p}(x,y) >=$$

$$=-\lim_{\lambda\to\lambda_s}\lim_{\mu\to\lambda_k}\frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\frac{1}{(m_k-1-p)!}\frac{\partial^{m_k-1-p}}{\partial\mu^{m_k-1-p}}\left(< \kappa_1(x,y,\lambda), M_1(x,y,\overline{\mu}) > \frac{(\mu-\lambda_k)^{m_k}}{d(\mu)}\right).$$

Recalling Lemma 2, we get equality

(6.1)
$$< W_{n,s}(x,y), h_{k,m_k-1-p}(x,y) >=$$

$$=\lim_{\lambda\to\lambda_s}\lim_{\mu\to\lambda_k}\frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\frac{1}{(m_k-1-p)!}\frac{\partial^{m_k-1-p}}{\partial\mu^{m_k-1-p}}\left(\frac{d(\lambda)-d(\mu)}{\lambda-\mu}\frac{(\mu-\lambda_k)^{m_k}}{d(\mu)}\right).$$

We introduce the notation

(6.2)
$$H_{k,t}(\lambda)=\lim_{\mu\to\lambda_k}\frac{1}{t!}\frac{\partial^t}{\partial\mu^t}\left(\frac{d(\lambda)-d(\mu)}{\lambda-\mu}\frac{(\mu-\lambda_k)^{m_k}}{d(\mu)}\right)$$

We consider the function

$$F(\mu)=\frac{d(\lambda)-d(\mu)}{\lambda-\mu}\frac{(\mu-\lambda_k)^{m_k}}{d(\mu)}$$

and expand it in a neighborhood of the point $\mu=\lambda_k$ into a Taylor series. Then

$$F(\mu)=H_{k,0}(\lambda)+H_{k,1}(\lambda)(\mu-\lambda_k)+\dots+H_{k,m_k-1}(\lambda)(\mu-\lambda_k)^{m_k-1}+\cdots,$$

that is, $H_{k,t}(\lambda)$ is the k-th Taylor coefficient, with a corresponding expansion, in a neighborhood of $\mu = \lambda_k$. The direct calculation of the coefficient of the Taylor series of the function $F(\mu)$ leads to the following formula for $t = 0,1,\dots,m_k - 1$:

(6.3)
$$H_{k,t}(\lambda) = d(\lambda)\left(A_{k,m_k-1}\frac{1}{(\lambda-\lambda_k)^{t+1}} + A_{k,m_k-2}\frac{1}{(\lambda-\lambda_k)^{t}} + \cdots + A_{k,m_k-t-1}\frac{1}{\lambda-\lambda_k}\right),$$

where number $A_{k,m_k-1},\dots,A_{k,0}$ define by identity

$$\frac{1}{d(\mu)} = \frac{A_{k,m_k-1}}{(\mu-\lambda_k)^{m_k}} + \frac{A_{k,m_k-2}}{(\mu-\lambda_k)^{m_k-1}} + \cdots + \frac{A_{k,0}}{\mu-\lambda_k} + \sum_{q=0}^{\infty} B_{k,q}(\mu-\lambda_k)^q.$$

If $\lambda_s \neq \lambda_k$, that of (6.1), (6.2) and (6.3) when $n = 0,1,\dots,m_k - 1$

$$< W_{n,s}(x,y), h_{k,m_k-1-p}(x,y) >= \lim_{\lambda\to\lambda_s}\frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}H_{k,t}(\lambda) =$$

$$= \lim_{\lambda\to\lambda_s}\frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}d(\lambda)\sum_{i=1}^{t+1}\frac{A_{k,m_k-t+i-2}}{(\lambda-\lambda_k)^i} = \frac{1}{n!}d^{(n)}(\lambda_s)\sum_{i=1}^{t+1}\frac{A_{k,m_k-t+i-2}}{(\lambda_s-\lambda_k)^i} = 0,$$

in that $d^{(n)}(\lambda_s) = 0$.

Now consider the case $\lambda_s = \lambda_k$. Transform the right side of the relation (6.3)

(6.4)
$$H_{k,t}(\lambda) = d(\lambda)\sum_{i=1}^{t+1}\frac{A_{k,m_k-t+i-2}}{(\lambda_s-\lambda_n)^i} =$$

$$= d(\lambda)(\lambda-\lambda_k)^{m_k-t-1}\left(A_{k,m_k-1}\frac{1}{(\lambda-\lambda_k)^{m_k}} + A_{k,m_k-2}\frac{1}{(\lambda-\lambda_k)^{m_k-1}} + \cdots + A_{k,m_k-1-t}\frac{1}{(\lambda-\lambda_k)^{m_k-t}}\right) = d(\lambda)(\lambda-\lambda_k)^{m_k-t-1}$$

$$\left(\frac{1}{d(\lambda)} - A_{k,m_k-2}\frac{1}{(\lambda-\lambda_k)^{m_k-1-t}} - \cdots - A_{k,0}\frac{1}{\lambda-\lambda_k} - \sum_{q=m_k}^{\infty} B_{k,q}(\lambda-\lambda_k)^q\right) =$$

$$(\lambda-\lambda_k)^{m_k-1-t} + \sum_{q=m_k}^{\infty} c_{kq}^{t}(\lambda-\lambda_k)^q,\ k = 0,1,\dots,m_k-1.$$

From relation (6.1), (6.2) and (6.3) we obtain

$$< W_{n,s}(x,y), h_{k,m_k-1-p}(x,y) >= \frac{1}{n!}\lim_{\lambda\to\lambda_k}\frac{\partial^n}{\partial\lambda^n}H_{k,m_k-1-p}(\lambda) =$$

$$\frac{1}{n!}\lim_{\lambda\to\lambda_k}\frac{\partial^n}{\partial\lambda^n}\left((\lambda-\lambda_k)^{m_k-1-t} + \sum_{q=m_k}^{\infty} c_{kq}^{t}(\lambda-\lambda_k)^q\right).$$

This implies the required assertion for $\lambda_s = \lambda_k$. Q.E.D.

Note the works of the authors [14, 15], where obtained the formulas of the first regularized traces for Laplace operator and double differentiation in the punctured areas.

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ОЙЫЛҒАН АЙМАҚТАҒЫ ЛАПЛАС ОПЕРАТОРЫНЫҢ ТҮБІР ФУНКЦИЯЛАР ЖҮЙЕСІНІҢ МИНИМАЛДЫЛЫҒЫ

Аннотация. Біз осы мақаламызда, қисынды шешілетін шеттік есептердің «жаңа» классын туындататын, ойылған аймақтағы Лаплас операторын қарастырамыз. Осы есеп үшін резольвента формуласы алынған. Сонымен қатар, зерттелген есептің түбір функцияларын туындататын мероморфты функциялар сипатталған. Негізгі мақсатымзы түбір функциялар жүйесінің минималдылығын қарастыру. Біздің жұмыс ойылған аймақтағы Лаплас операторы үшін шеттік есептің шешімділігі сипатталған [8] жұмысының жалғасы болып табылады. Мақалада қисынды шешілетін шеттік есептердің «жаңа» классын туындататын, ойылған аймақтағы Лаплас операторы қарастырылып, туындаған есеп үшін резольвента формуласы алынған, сонымен қатар, функциялар жүйесін құратын мероморфты функциялар сипатталған. Меншікті және қосалқы функциялардың жүйесі осындай жүйенің бірі болып табылады. Соңғы бөлім түбір функциялардың минималдылығын зерттеуге арналған.

Түйін сөздер: Лаплас операторы, ойылған аймақ, резольвента, мероморфты функция, шеттік есептің қисынды шешімділігі, түбір функциялар жүйесі, минимал жүйе.

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О МИНИМАЛЬНОСТИ СИСТЕМ КОРНЕВЫХ ФУНКЦИЙ ОПЕРАТОРА ЛАПЛАСА В ПРОКОЛОТОЙ ОБЛАСТИ

Аннотация. В данной работе рассмотрен оператор Лапласа в проколотой области, который порождает класс "новых" , корректно разрешимых краевых задач. И для этого класса задач получена формула резольвенты. Также описаны мероморфные функции, порождающие корневых функций класса исследуемых задач. Основная цель – изучение минимальности систем корневых функций. Статья является продолжением работы [8], где дано описание корректно разрешимых краевых задач для оператора Лапласа в проколотых областях. Рассмотрен оператор Лапласа в проколотой области, который порождает класс "новых" , корректно разрешимых краевых задач, и для порожденных задач получена формула резольвенты, а также описаны мероморфные функции, которые индуцируют системы функций. Одна из этих систем, как раз, и является системой собственных и присоединенных функций. Последний раздел посвящен исследованию минимальности системы корневых функций.

Ключевые слова: Оператор Лапласа, проколотая область, резольвента, мероморфная функция, корректно разрешимая краевая задача, система корневых функций, минимальная система.

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