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INVERSE PROBLEM OF THE STURM-LIOUVILLE OPERATOR WITH UNSEPARATED BOUNDARY VALUE CONDITIONS AND SYMMETRIC POTENTIAL

Abstract. In the paper we prove the uniqueness theorem on a single spectrum for the Sturm-Liouville operator with unseparated boundary value conditions and real continuous symmetric potential. The research method is different from all known methods, and is based on the internal symmetry of the operator generated by invariant subspaces.
Keywords: Sturm - Liouville operator, spectrum, Sturm - Liouville inverse problem, Borg theorem, Ambartsumian theorem, Levinson theorem, unseparated boundary value conditions, symmetric potential, invariant subspaces.

1. Introduction. By inverse problems of spectral analysis, we understand the problems about restoration of a linear operator according to one or another of its spectral characteristics.

The first significant result in this direction was obtained in 1929 by V.A. Ambartsumyan [1]. He proved the following theorem.

By $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ we denote eigenvalues of the following Sturm-Liouville problem

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$y'(0) = 0,\ y'(\pi) = 0; \tag{1.2}$$

where $q(x)$ is a real continuous function. If

$$\lambda_n = n^2\ (n = 0,1,2,\dots)\ \text{ then } q(x) \equiv 0.$$

The first mathematician who drew attention to importance of the V.A.Ambartsumian's result was the Swedish mathematician Borg. He also performed the first systematic study of one of the important inverse problems, namely, inverse problem for the classical Sturm-Liouville operator of the form (1.1) by spectra [2]. Borg showed that, in general case, one spectrum of the Sturm-Liouville operator does not define it, so Ambartsumian's result is an exception from the general rule. In the same paper [2], Borg shows that two spectra of the Sturm – Liouville operator (with different boundary conditions) uniquely determine it. More precisely, Borg proved the following theorem.

Borg Theorem.

Suppose that the following equations

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$-z'' + p(x)z = \lambda z, \tag{1.3}$$

have the same spectrum under the boundary value conditions

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma y(\pi) + \delta y'(\pi) = 0; \end{cases} \tag{1.4}$$

and under the boundary value conditions

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma' y(\pi) + \delta' y'(\pi) = 0. \end{cases} \tag{1.4$'$}$$

Then $q(x) = p(x)$ almost everywhere on the segment $[0, \pi]$, if

$$\delta \cdot \delta' = 0, \qquad |\delta| + |\delta'| > 0.$$

Shortly after Borg's work, important studies on the theory of inverse problems were performed by Levinson [3], in particular, he proved that if $q(\pi - x) = q(x)$, then the Sturm-Liouville operator

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0 \end{cases} \tag{1.5}$$

is reconstructed by one spectrum.

A number of works by B.M. Levitan [4], [5] are devoted to reconstructing the Sturm-Liouville operator by one and two spectra. These studies have found continuations in [6] - [20]. Sources [21] - [28] are introductory.

This paper is devoted to generalization of Ambartsumian [1] and Levinson [3] theorems, in particular, our results contain the results of these authors.

2. Research Methods.

Idea of this work is very simple. Analysis of contents of [1, 3] showed that both of these operators have an invariant subspace. If for a linear operator L the following formulas hold

$$LP = PL^*, \qquad QL = L^*Q,$$

where P, Q are orthogonal projections, satisfying the condition $P + Q = I$, then the operators L and L^* have invariant subspaces, sometimes restriction of these operators to these invariant subspaces, under certain conditions, form a Borg pair.

3. Research Results.

In the Hilbert space $H = L^2(0, \pi)$ we consider the Sturm-Liouville operator

$$Ly = -y'' + q(x)y; \tag{3.1}$$

$$\begin{cases} a_{11}y(0) + a_{12}y'(0) + a_{13}y(\pi) + a_{14}y'(\pi) = 0, \\ a_{21}y(0) + a_{22}y'(0) + a_{23}y(\pi) + a_{24}y'(\pi) = 0 \end{cases} \tag{3.2}$$

where $q(x)$ is a continuous complex function, a_{ij} $(i = 1,2;\ j = 1,2,3,4)$ are arbitrary complex coefficients, and by Δ_{ij} $(i = 1,2;\ j = 1,2,3,4)$ we denote the minors of the boundary matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

Assume that $a_{24} \neq 0$, then the Sturm-Liouville operator (3.1) – (3.2) takes the following form

$$Ly = -y'' + q(x)y,\ x \in (0, \pi); \tag{3.1}$$

$$\begin{cases} \Delta_{14}y(0) + \Delta_{24}y'(0) + \Delta_{34}y(\pi) = 0, \\ \Delta_{12}y(0) + \Delta_{32}y(\pi) - \Delta_{24}y'(\pi) = 0 \end{cases} \tag{3.3}$$

and its adjoint operator L^+ takes the form

$$L^+z = -z'' + \overline{q(x)}z,\ x \in (0, \pi); \tag{3.1$^+$}$$

$$\begin{cases}\overline{\Delta_{14}}z(0)+\overline{\Delta_{24}}z'(0)+\overline{\Delta_{12}}z(\pi)=0,\\ \overline{\Delta_{34}}z(0)+\overline{\Delta_{32}}z(\pi)-\overline{\Delta_{24}}z'(\pi)=0.\end{cases} \qquad (3.3)^{+}$$

Let P and Q be orthogonal projections, defined by the formulas

$$Pu(x)=\frac{u(x)+u(\pi-x)}{2},\ Qv(x)=\frac{v(x)-v(\pi-x)}{2} \qquad (3.4)$$

The main result of the paper is the following theorem.

Theorem 3.1. If $\Delta_{24}\neq 0$, and

1) $PL=L^{+}P$;
2) $LQ=QL^{+}$;
3) $\Delta_{12}=-\Delta_{34}$;

then the Sturm-Liouville operator (3.1) – (3.3) reconstructed by one spectrum.

4. Discussion.

In this section we prove the theorem and discuss the obtained results. Following Lemmas 4.1, 4.2 may have independent meanings.

Lemma 4.1. If for a linear and discrete operator L the following equalities hold

1) $PL=L^{+}P$;
2) $LQ=QL^{+}$;
3) $P+Q=I$;

where P,Q are orthogonal projections, and I is identity operator, then all its eigenvalues are real.

Proof.

Let $PL=L^{+}P$, $LQ=QL^{+}$; then

$$(PL)^{*}=L^{*}P^{*}=L^{*}P=PL;$$

$$(LQ)^{*}=Q^{*}L^{*}=QL^{*}=LQ;$$

i.e. operators PL and LQ are self-adjoint, therefore, their eigenvalues are real.

Further, if $Ly=\lambda y,\ y\neq 0$, then $PLy=\lambda Py,\ L^{+}Py=\lambda Py,\ L^{+}P(Py)=\lambda Py$, consequently, if $Py\neq 0$, then λ is a real quantity; if $Py=0$, then $y=Qy$, and $LQy=\lambda Qy,\ LQ(Qy)=\lambda Qy,\ Qy=y\neq 0$. Thus, λ is again a real quantity.

The following lemma shows that the spectrum $\sigma(L)$ of the operator L consists of two parts; therefore, the operator L, apparently, splits into two parts. Later we will see that this is exactly what happens, and moreover, these parts form a Borg pair under a certain condition.

Lemma 4.2. If L is a linear discrete operator, satisfying the conditions:

1) $PL=L^{+}P$;
2) $LQ=QL^{+}$;
3) $P+Q=I$;

where P,Q are orthogonal projections, and I is identity operator, then

$$\sigma(L)=\sigma(L_1)\cup\sigma(L_2).$$

where $\sigma(L)$ is a spectrum of the operator L, $L_1=PL,\ L_2=LQ$.

Proof.

If $Ly=\lambda y,\ y\neq 0$, then $PLy=\lambda Py,\ L^{+}Py=\lambda Py,\ L^{+}P(Py)=\lambda Py,\ L_1(Py)=\lambda Py$. If $Py\neq 0$, then $\lambda\in\sigma(L_1)$. If $Py=0$, then $y=Qy\neq 0$ and $LQy=\lambda Qy,\ LQ(Qy)=\lambda Qy,\ L_2Qy=\lambda Qy$. Consequently, $\lambda\in\sigma(L_2)$.

Thus $\sigma(L)\subset\sigma(L_1)\cup\sigma(L_2)$, where $\sigma(A)$ means spectrum of the operator A.

Assume that $\lambda\in\sigma(L_1)\cup\sigma(L_2)$ and $\lambda\neq 0$. Then

a) If $\lambda\in\sigma(L_1)$, then $PLy=\lambda y,\ y\neq 0\ \to\ P^2Ly=\lambda Py, PLy=\lambda Py$, if $Py=0$, then $PLy=0,=>\lambda y=0,\to y=0$, it is impossible, therefore $Py\neq 0$ and $L^{*}Py=\lambda Py$. Consequently, $\lambda\in\sigma(L^{*})=\sigma(L)$.

b) If $\lambda\in\sigma(L_2)$, then $LQy=\lambda y,\ y\neq 0,\ QL^{*}y=\lambda y,\ Q^2L^{*}y=\lambda Qy,\ QL^{*}y=\lambda Qy,\to Qy\neq 0$, otherwise $QL^{*}y=0,=>LQy=0,\to\lambda y=0,\to y=0$. Consequently, $\lambda Qy=QL^{*}y=LQy,\to L(Qy)=\lambda Qy$, thus $\lambda\in\sigma(L)$.

c) $0\in\sigma(L_1)\cup\sigma(L_2)$.

If $0 \in \sigma(L_1)$, then $L_1 u = 0,\ PLu = 0,\ u \neq 0$; Operator L_1 maps the subspace $H_1 = PH$ into H_1, thus $u = Pu \neq 0$, then $PLu = L^* Pu = L^* u = 0, \Rightarrow 0 \in \sigma(L^*) = \sigma(L)$.

If $0 \in \sigma(L_2)$, then $L_2 v = 0,\ v \neq 0$; Operator L_2 maps the subspace $H_2 = QH$ into H_2, therefore $v = Qv$ and $L_2 v = LQv = Lv = 0,\ \Rightarrow 0 \in \sigma(L)$.

Lemma 4.3. If

a) $\Delta_{24} \neq 0$; (4.1)

b) $PL = L^+ P$; (4.2)

where $Pu(x) = \frac{u(x)+u(\pi-x)}{2}$, $Qv(x) = \frac{v(x)-v(\pi-x)}{2}$, then for the Sturm - Liouville operator (3.1) – (3.2) the following equalities hold

1) $q(\pi - x) = q(x)$;

2) $\overline{q(x)} = q(x)$;

3) $\Delta_{12} + \Delta_{14} = \Delta_{32} + \Delta_{34}$;

4) $\overline{\left(\frac{\Delta_{12}+\Delta_{14}}{\Delta_{24}}\right)} = \frac{\Delta_{12}+\Delta_{14}}{\Delta_{24}} = \frac{\Delta_{32}+\Delta_{34}}{\Delta_{24}}$;

Moreover, operators L and L^+ take the following forms:

$Ly = -y'' + q(x)y,\ x \in (0,\pi)$;

$$\begin{cases} \frac{\Delta_{12}+\Delta_{14}}{\Delta_{24}}[y(0) + y(\pi)] + y'(0) - y'(\pi) = 0, \\ \Delta_{12} y(0) + \Delta_{32} y(\pi) - \Delta_{24} y'(\pi) = 0. \end{cases}$$

$L^+ z = -z'' + q(x)z,\ x \in (0,\pi)$;

$$\begin{cases} \frac{\overline{\Delta_{14}}-\overline{\Delta_{34}}}{\overline{\Delta_{24}}}[z(0) + z(\pi)] + z'(0) + z'(\pi) = 0, \\ \overline{\Delta_{14}} z(0) + \overline{\Delta_{24}} z'(0) + \overline{\Delta_{12}} z(\pi) = 0. \end{cases}$$

Proof.

If $PL = L^+ P$, then $z(x) = Py(x) \in D(L^+)$, where $y(x) \in D(L)$, thus

$$\begin{cases} \overline{\Delta_{14}}\dfrac{y(0)+y(\pi)}{2} + \overline{\Delta_{24}}\dfrac{y'(0)-y'(\pi)}{2} + \overline{\Delta_{12}}\dfrac{y(\pi)+y(0)}{2} = 0, \\ \overline{\Delta_{34}}\dfrac{y(0)+y(\pi)}{2} + \overline{\Delta_{32}}\dfrac{y(\pi)+y(0)}{2} - \overline{\Delta_{24}}\dfrac{y'(\pi)-y'(0)}{2} = 0; \end{cases}$$

$$\begin{cases} (\overline{\Delta_{12}} + \overline{\Delta_{14}})[y(0) + y(\pi)] + \overline{\Delta_{24}}[y'(0) - y'(\pi)] = 0, \\ (\overline{\Delta_{32}} + \overline{\Delta_{34}})[y(0) + y(\pi)] + \Delta_{24}[y'(0) - y'(\pi)] = 0. \end{cases}$$

For unknown quantities $y(0), y'(0), y(1), y'(1)$ we obtained the system of equations, therefore

$$\begin{vmatrix} \Delta_{14} & \Delta_{24} & \Delta_{34} & 0 \\ \Delta_{12} & 0 & \Delta_{32} & -\Delta_{24} \\ \overline{\Delta_{12}} + \overline{\Delta_{14}} & \overline{\Delta_{24}} & \overline{\Delta_{12}} + \overline{\Delta_{14}} & -\Delta_{24} \\ \overline{\Delta_{32}} + \overline{\Delta_{34}} & \Delta_{24} & \overline{\Delta_{32}} + \overline{\Delta_{34}} & -\Delta_{24} \end{vmatrix} = 0, \Rightarrow$$

$$\begin{vmatrix} \Delta_{14} & \Delta_{24} & \Delta_{34} & \Delta_{24} \\ \Delta_{12} & 0 & \Delta_{32} & -\Delta_{24} \\ \overline{\Delta_{12}} + \overline{\Delta_{14}} & \overline{\Delta_{24}} & \overline{\Delta_{12}} + \overline{\Delta_{14}} & 0 \\ \overline{\Delta_{32}} + \overline{\Delta_{34}} & \overline{\Delta_{24}} & \overline{\Delta_{32}} + \overline{\Delta_{34}} & 0 \end{vmatrix} \sim \begin{vmatrix} \Delta_{14} & \Delta_{24} & \Delta_{34} & \Delta_{24} \\ \Delta_{12} + \Delta_{14} & \Delta_{24} & \Delta_{32} + \Delta_{34} & 0 \\ \overline{\Delta_{12}} + \overline{\Delta_{14}} & \overline{\Delta_{24}} & \overline{\Delta_{12}} + \overline{\Delta_{14}} & 0 \\ \overline{\Delta_{32}} + \overline{\Delta_{34}} & \overline{\Delta_{24}} & \overline{\Delta_{32}} + \overline{\Delta_{34}} & 0 \end{vmatrix} =$$

$$= (-\Delta_{24})\begin{vmatrix} \Delta_{12}+\Delta_{14} & \Delta_{24} & \Delta_{32}+\Delta_{34} \\ \overline{\Delta_{12}}+\overline{\Delta_{14}} & \overline{\Delta_{24}} & \overline{\Delta_{12}}+\overline{\Delta_{14}} \\ \overline{\Delta_{32}}+\overline{\Delta_{34}} & \overline{\Delta_{24}} & \overline{\Delta_{32}}+\overline{\Delta_{34}} \end{vmatrix} =$$

$$= (-\Delta_{24})\begin{vmatrix} \Delta_{12}+\Delta_{14} & \Delta_{24} & \Delta_{32}+\Delta_{34}-\Delta_{12}-\Delta_{14} \\ \overline{\Delta_{12}}+\overline{\Delta_{14}} & \overline{\Delta_{24}} & 0 \\ \overline{\Delta_{32}}+\overline{\Delta_{34}} & \overline{\Delta_{24}} & 0 \end{vmatrix} =$$

$$= (-\Delta_{24})(\Delta_{32}+\Delta_{34}-\Delta_{12}-\Delta_{14})\overline{\Delta_{24}}(\overline{\Delta_{12}}+\overline{\Delta_{14}}-\overline{\Delta_{32}}-\overline{\Delta_{34}}) =$$

$$= |\Delta_{24}|^2|\Delta_{32}+\Delta_{34}-\Delta_{12}-\Delta_{14}|^2 = 0.$$

Since $\Delta_{24} \neq 0$, then $\Delta_{12}+\Delta_{14} = \Delta_{32}+\Delta_{34}$.

Summing up the boundary conditions (3.3), we have

$$(\Delta_{12}+\Delta_{14})y(0)+(\Delta_{32}+\Delta_{34})y(\pi)+\Delta_{24}[y'(0)-y'(\pi)] = 0,$$

$$(\Delta_{12}+\Delta_{14})[y(0)+y(\pi)]+\Delta_{24}[y'(0)-y'(\pi)] = 0.$$

Combining this equation with the first boundary condition (4.3), we receive

$$\begin{cases} (\Delta_{12}+\Delta_{14})[y(0)+y(\pi)]+\Delta_{24}[y'(0)-y'(\pi)] = 0, \\ (\overline{\Delta_{12}}+\overline{\Delta_{14}})[y(0)+y(\pi)]+\overline{\Delta_{24}}[y'(0)-y'(\pi)] = 0. \end{cases} \tag{4.5}$$

This system of equations (4.5) has a non-trivial solution, therefore

$$\Delta = \begin{vmatrix} \Delta_{12}+\Delta_{14} & \Delta_{24} \\ \overline{\Delta_{12}}+\overline{\Delta_{14}} & \overline{\Delta_{24}} \end{vmatrix} = 0,$$

$$\overline{\Delta_{24}}(\Delta_{12}+\Delta_{14})-\Delta_{24}(\overline{\Delta_{12}}+\overline{\Delta_{14}}) = 0,$$

$$\frac{\overline{\Delta_{12}}+\overline{\Delta_{14}}}{\overline{\Delta_{24}}} = \frac{\Delta_{12}+\Delta_{14}}{\Delta_{24}}.$$

Hence, the operator L has the following form

$$Ly = -y''+q(x)y, \qquad x \in (0,\pi);$$

$$\begin{cases} \dfrac{\Delta_{12}+\Delta_{14}}{\Delta_{24}}[y(0)+y(\pi)]+y'(0)-y'(\pi) = 0, \\ \Delta_{12}y(0)+\Delta_{32}y(\pi)-\Delta_{24}y'(\pi) = 0. \end{cases}$$

We specify the boundary conditions of the operator L^{+}, subtracting the second boundary condition from the first boundary condition $(3.3)^{+}$, we get

$$(\overline{\Delta_{14}}-\overline{\Delta_{34}})z(0)+(\overline{\Delta_{12}}-\overline{\Delta_{32}})z(\pi)+\overline{\Delta_{24}}[z'(0)+z'(\pi)] = 0.$$

From the formula 4.4 it follows that $\overline{\Delta_{14}}-\overline{\Delta_{34}} = \overline{\Delta_{32}}-\overline{\Delta_{12}}$, thus

$$(\overline{\Delta_{14}}-\overline{\Delta_{34}})[z(0)-z(\pi)]+\overline{\Delta_{24}}[z'(0)+z'(\pi)] = 0,$$

$$\frac{\overline{\Delta_{14}}-\overline{\Delta_{34}}}{\overline{\Delta_{24}}}[z(0)-z(\pi)]+z'(0)+z'(\pi)=0.$$

Consequently, the adjoint operator L^{+} has the following form

$$L^{+}z=-z''+q(x)z,$$

$$\begin{cases}\dfrac{\overline{\Delta_{14}}-\overline{\Delta_{34}}}{\overline{\Delta_{24}}}[z(0)-z(\pi)]+z'(0)+z'(\pi)=0,\\ \overline{\Delta_{14}}z(0)+\overline{\Delta_{24}}z'(0)+\overline{\Delta_{12}}z(\pi)=0;\end{cases}$$

Further, the formula $PL=L^{+}P$ implies

$$PLy=P^{\circ}[-y''+q(x)y]=-P[y''(x)]+P[q(x)y(x)]=$$

$$=-\frac{y''(x)+y''(\pi-x)}{2}+\frac{q(x)y(x)+q(\pi-x)y(\pi-x)}{2};$$

$$L^{+}Py=L^{+}\left[\frac{y(x)+y(\pi-x)}{2}\right]=$$

$$=-\frac{y''(x)+y''(\pi-x)}{2}+\bar{q}(x)\frac{y(x)+y(\pi-x)}{2};$$

$$q(x)y(x)+q(\pi-x)y(\pi-x)=\bar{q}(x)y(x)+\bar{q}(x)y(\pi-x),$$

$$[q(x)-\bar{q}(x)]y(x)+[q(\pi-x)-\bar{q}(x)]y(\pi-x)=0,=> \qquad (4.6)$$

$$[q(\pi-x)-\bar{q}(\pi-x)]y(\pi-x)+[q(x)-\bar{q}(\pi-x)]y(x)=0.$$

It is obvious that the system of equations for $y(x)$ and $y(\pi-x)$ has a non-trivial solution, thus

$$\Delta=\begin{vmatrix} q(x)-\bar{q}(x) & q(\pi-x)-\bar{q}(x)\\ q(x)-\bar{q}(\pi-x) & q(\pi-x)-\bar{q}(\pi-x)\end{vmatrix}=0;$$

$$[q(x)-\bar{q}(x)][q(\pi-x)-\bar{q}(\pi-x)]=[q(\pi-x)-\bar{q}(x)][q(x)-\bar{q}(\pi-x)],$$

$$q(x)q(\pi-x)-q(x)\,\bar{q}(\pi-x)-\bar{q}(x)\,q(\pi-x)+\bar{q}(x)\bar{q}(\pi-x)=$$

$$=q(\pi-x)q(x)-q(\pi-x)\bar{q}(\pi-x)-q(x)\bar{q}(x)-\bar{q}(x)\bar{q}(\pi-x),$$

$$q(x)\bar{q}(\pi-x)+\bar{q}(x)\,q(\pi-x)=q(\pi-x)\bar{q}(\pi-x)+q(x)\bar{q}(x),$$

$$q(x)[\bar{q}(\pi-x)-\bar{q}(x)]+q(\pi-x)[\bar{q}(x)-\bar{q}(\pi-x)]=0,$$

$$[\bar{q}(x)-\bar{q}(\pi-x)][q(\pi-x)-q(x)]=0,$$

$$|q(x)-q(\pi-x)|^{2}=0,=> q(x)-q(\pi-x)=0; \qquad (4.7)$$

From the formulas (4.6) and (4.7), we have

$$[q(x)-\bar{q}(x)]y(x)+[q(x)-\bar{q}(x)]y(\pi-x)=0,=>$$

$$[q(x)-\bar{q}(x)][y(x)+y(\pi-x)]=0,=> q(x)-\bar{q}(x)=0.$$

Lemma 4.4. If

a) $\Delta_{24} \neq 0$;

б) $LQ = QL^{+}$,

then

1) $q(\pi - x) = q(x)$;
2) $\bar{q}(x) = q(x)$,
3) $\overline{\Delta_{12} + \Delta_{14}} = \Delta_{32} + \Delta_{34}$;
4) $\underbrace{\overline{\left(\frac{\Delta_{14}-\Delta_{34}}{\Delta_{24}}\right)}}_{\frac{\Delta_{32}-\Delta_{12}}{\Delta_{24}}} = \left(\frac{\Delta_{14}-\Delta_{34}}{\Delta_{24}}\right) = \frac{\Delta_{32}-\Delta_{12}}{\Delta_{24}}$,

where $Qz = \frac{z(x)-z(\pi-x)}{2}$.

Moreover, the operators L and L^{+} take the following forms

$$Ly = -y'' + q(x)y, \ x \in (0,\pi);$$

$$\begin{cases} \frac{\Delta_{12}+\Delta_{14}}{\Delta_{24}}[y(0) + y(\pi)] + y'(0) - y'(\pi) = 0, \\ \Delta_{12}y(0) + \Delta_{32}y(\pi) - \Delta_{24}y'(\pi) = 0; \end{cases}$$

$$L^{+}z = -z'' + q(x)z, \ x \in (0,\pi);$$

$$\begin{cases} \frac{\Delta_{14}-\Delta_{34}}{\Delta_{24}}[z(0) - z(\pi)] + z'(0) + z'(\pi) = 0, \\ \overline{\Delta_{14}}z(0) + \overline{\Delta_{24}}z'(0) + \overline{\Delta_{12}}z(\pi) = 0. \end{cases}$$

Proof.

If $LQ = QL^{+}$ and $z \in D(L^{+})$, then $y = Qz \in D(L)$, therefore

$$y(x) = \frac{z(x) - z(\pi - x)}{2}, \qquad y'(x) = \frac{z'(x) + z'(\pi - x)}{2};$$

$$\begin{cases} \Delta_{14}\frac{z(0) - z(\pi)}{2} + \Delta_{24}\frac{z'(\pi) + z'(0)}{2} + \Delta_{34}\frac{z(\pi) - z(0)}{2} = 0, \\ \Delta_{12}\frac{z(0) - z(\pi)}{2} + \Delta_{32}\frac{z(\pi) + z(0)}{2} - \Delta_{24}\frac{z'(\pi) + z'(0)}{2} = 0; \end{cases}$$

$$\begin{cases} (\Delta_{14} - \Delta_{34})\frac{z(0)-z(\pi)}{2} + \Delta_{24}\frac{z'(0)+z'(\pi)}{2} = 0, \\ (\Delta_{12} - \Delta_{32})\frac{z(0)-z(\pi)}{2} - \Delta_{24}\frac{z'(0)+z'(\pi)}{2} = 0; \end{cases} \tag{4.8}$$

It is obvious that the system has a non-trivial solution, therefore

$$\Delta = \begin{vmatrix} \Delta_{14} - \Delta_{34} & \Delta_{24} \\ \Delta_{12} - \Delta_{32} & -\Delta_{24} \end{vmatrix} = -\Delta_{24}(\Delta_{14} - \Delta_{34} + \Delta_{12} - \Delta_{32}) = 0,$$

since $\Delta_{24} \neq 0$, then $\Delta_{12} + \Delta_{14} = \Delta_{32} + \Delta_{34}$. (4.9)

Subtracting the second boundary condition from the first boundary condition (3.3)[+], we get

$$(\overline{\Delta_{14}} - \overline{\Delta_{34}})z(0) + (\overline{\Delta_{12}} - \overline{\Delta_{32}})z(\pi) + \overline{\Delta_{24}}[z'(0) + z'(\pi)] = 0$$

Due to the formula (4.9), this formula takes the form

$$(\overline{\Delta_{14}} - \overline{\Delta_{34}})z(0) - (\overline{\Delta_{14}} - \overline{\Delta_{34}})z(\pi) + \overline{\Delta_{24}}[z'(0) + z'(\pi)] = 0,$$

$$\frac{\overline{\Delta_{14}} - \overline{\Delta_{34}}}{\overline{\Delta_{24}}}[z(0) - z(\pi)] + z'(0) + z'(\pi) = 0.$$

Combining this boundary condition with the first boundary condition (4.8), we obtain

$$\begin{cases} \dfrac{\Delta_{14} - \Delta_{34}}{\Delta_{24}}[z(0) - z(\pi)] + z'(0) + z'(\pi) = 0, \\ \dfrac{\overline{\Delta_{14}} - \overline{\Delta_{34}}}{\overline{\Delta_{24}}}[z(0) - z(\pi)] + z'(0) + z'(\pi) = 0. \end{cases}$$

$$\Delta = \begin{vmatrix} \dfrac{\Delta_{14} - \Delta_{34}}{\Delta_{24}} & 1 \\ \dfrac{\overline{\Delta_{14}} - \overline{\Delta_{34}}}{\overline{\Delta_{24}}} & 1 \end{vmatrix} = 0, \qquad \overline{\left(\frac{\Delta_{14} - \Delta_{34}}{\Delta_{24}}\right)} = \frac{\Delta_{14} - \Delta_{34}}{\Delta_{24}}.$$

Hence, the operator L^+ has the form

$$L^+z = -z'' + \bar{q}(x)z, \qquad x \in (0, \pi);$$

$$\begin{cases} \dfrac{\Delta_{14} - \Delta_{34}}{\Delta_{24}}[z(0) - z(\pi)] + z'(0) + z'(\pi) = 0, \\ \overline{\Delta_{14}}z(0) + \overline{\Delta_{24}}z'(0) + \overline{\Delta_{12}}z(\pi) = 0; \end{cases}$$

where $\frac{\Delta_{14} - \Delta_{34}}{\Delta_{24}}$ is a real quantity.

Summing up the boundary conditions (3.3), we get

$$(\Delta_{12} + \Delta_{14})y(0) + (\Delta_{32} + \Delta_{34})y(\pi) + \Delta_{24}[y'(0) - y'(\pi)] = 0,$$

$$\frac{\Delta_{12} + \Delta_{14}}{\Delta_{24}}[y(0) + y(\pi)] + y'(0) - y'(\pi) = 0.$$

Consequently, the operator L has the form

$$Ly = -y'' + q(x)y, \qquad x \in (0, \pi);$$

$$\begin{cases} \dfrac{\Delta_{12} + \Delta_{14}}{\Delta_{24}}[y(0) + y(\pi)] + y'(0) - y'(\pi) = 0, \\ \Delta_{12}y(0) + \Delta_{32}y(\pi) - \Delta_{24}y'(\pi) = 0. \end{cases}$$

Further, from the formula $LQ = QL^+$, we have

$$LQz = L^\circ \frac{z(x) - z(\pi - x)}{2} = -\frac{z''(x) - z''(\pi - x)}{2} + q(x)\frac{z(x) - z(\pi - x)}{2};$$

$$QL^+z = Q^\circ\left[-z'' + \overline{q(x)}z\right] = -\frac{z''(x) - z''(\pi - x)}{2} +$$

$$+\frac{\bar{q}(x)z(x) - \bar{q}(\pi - x)z(\pi - x)}{2};$$

$$q(x)z(x) - q(x)z(\pi - x) = \bar{q}(x)z(x) - \bar{q}(\pi - x)z(\pi - x),$$

$$\begin{cases} [q(x) - \bar{q}(x)]z(x) + [\bar{q}(\pi - x) - q(x)]z(\pi - x) = 0, \\ [\bar{q}(x) - q(\pi - x)]z(x) + [q(\pi - x) - \bar{q}(\pi - x)]z(\pi - x) = 0; \end{cases} \tag{4.10}$$

$$\Delta=\begin{vmatrix} q(x)-\bar q(x) & \bar q(\pi-x)-q(x) \\ \bar q(x)-q(\pi-x) & q(\pi-x)-\bar q(\pi-x)\end{vmatrix}=0;$$

$$[q(x)-\bar q(x)]\cdot[q(\pi-x)-\bar q(\pi-x)]=[\bar q(\pi-x)-q(x)]\cdot[\bar q(x)-q(\pi-x)]$$

$$q(x)q(\pi-x)-q(x)\bar q(\pi-x)-\bar q(x)q(\pi-x)+\bar q(x)\bar q(\pi-x)=$$

$$=\bar q(\pi-x)\bar q(x)-\bar q(\pi-x)q(\pi-x)+q(x)\bar q(x)+q(x)q(\pi-x);$$

$$q(x)\bar q(\pi-x)+\bar q(x)q(\pi-x)=\bar q(\pi-x)q(\pi-x)+q(x)\bar q(x),$$

$$q(x)[\bar q(\pi-x)-\bar q(x)]+q(\pi-x)[\bar q(x)-\bar q(\pi-x)]=0,$$

$$[\bar q(x)-\bar q(\pi-x)]\cdot[q(\pi-x)-q(x)]=0,$$

$$|q(x)-q(\pi-x)|^2=0,\ \Rightarrow\ q(x)=q(\pi-x).$$

From (4.10) it follows that

$$[q(x)-\bar q(x)]z(x)+[\bar q(x)-q(x)]z(\pi-x)=0,\Rightarrow$$

$$[q(x)-\bar q(x)][z(x)-z(\pi-x)]=0,\Rightarrow\ q(x)=\bar q(x).$$

The proved Lemmas 4.3 and 4.4 yields the following theorem.

Theorem 4.1. If $\Delta_{24}\neq 0$ and the following formulas hold

a) $PL=L^{+}P$,

b) $LQ=QL^{+}$;

then the operators L and L^{+} take the following forms:

$$Ly=-y''+q(x)y,\ x\in(0,\pi);$$

$$\begin{cases}\frac{\Delta_{12}+\Delta_{14}}{\Delta_{24}}[y(0)+y(\pi)]+y'(0)-y'(\pi)=0,\\ \Delta_{12}y(0)+\Delta_{32}y(\pi)-\Delta_{24}y'(\pi)=0;\end{cases}$$

$$L^{+}z=-z''+q(x)z,\ x\in(0,\pi);$$

$$\begin{cases}\frac{\Delta_{14}-\Delta_{34}}{\Delta_{24}}[z(0)-z(\pi)]+z'(0)+z'(\pi)=0,\\ \overline{\Delta_{14}}z(0)+\overline{\Delta_{24}}z'(0)+\overline{\Delta_{12}}z(\pi)=0;\end{cases}$$

where

1) $q(\pi-x)=q(x)$;
2) $\overline{q(x)}=q(x)$,
3) $\overline{\left(\frac{\Delta_{12}+\Delta_{14}}{\Delta_{24}}\right)}=\frac{\Delta_{12}+\Delta_{14}}{\Delta_{24}}=\frac{\Delta_{32}+\Delta_{34}}{\Delta_{24}}$;
4) $\frac{\Delta_{14}-\Delta_{34}}{\Delta_{24}}=\overline{\left(\frac{\Delta_{14}-\Delta_{34}}{\Delta_{24}}\right)}=\frac{\Delta_{32}-\Delta_{12}}{\Delta_{24}}$.

Further, from the formula $PL=L^{+}P$ we note that the operator $L_1=PL$ acts in the subspace $H_1=PH$, where $H=L^2(0,\pi)$. Supposing,

$$u(x)=Py(x)=\frac{y(x)+y(\pi-x)}{2},$$

we receive

$$u'(x)=\frac{y'(x)-y'(\pi-x)}{2}.$$

Then Theorem 4.1 implies

$$\begin{cases} u'(0) + \dfrac{\Delta_{12} + \Delta_{14}}{\Delta_{24}} u(0) = 0, \\ u'\left(\dfrac{\pi}{2}\right) = 0; \end{cases}$$

$$L_1 u = -u'' + q(x)u, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} u'(0) + \dfrac{\Delta_{12} + \Delta_{14}}{\Delta_{24}} u(0) = 0, \\ u'\left(\dfrac{\pi}{2}\right) = 0. \end{cases}$$

Similarly, supposing

$$v(x) = \frac{z(x) - z(\pi - x)}{2},$$

we get

$$v'(x) = \frac{z'(x) + z'(\pi - x)}{2}.$$

Then from Theorem 4.1 it follows that

$$\begin{cases} v'(0) + \dfrac{\Delta_{14} - \Delta_{34}}{\Delta_{24}} v(0) = 0, \\ v\left(\dfrac{\pi}{2}\right) = 0; \end{cases}$$

$$L_2 v = -v'' + q(x)v, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} v'(0) + \dfrac{\Delta_{14} - \Delta_{34}}{\Delta_{24}} v(0) = 0, \\ v\left(\dfrac{\pi}{2}\right) = 0; \end{cases}$$

From the condition 3) of the proved Theorem 4.1 it follows the following equality

$$\Delta_{12} + \Delta_{14} = \Delta_{14} - \Delta_{34}$$

Supposing $\alpha = \Delta_{24}$, $\beta = \Delta_{12} + \Delta_{14}$, we rewrite the operators L_1 and L_2 in the form

$$L_1 u = -u'' + q(x)u, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \alpha u'(0) + \beta u(0) = 0, \\ u'\left(\dfrac{\pi}{2}\right) = 0; \end{cases}$$

$$L_2 v = -v'' + q(x)v, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \alpha v'(0) + \beta v(0) = 0, \\ v\left(\dfrac{\pi}{2}\right) = 0. \end{cases}$$

If spectrum of the operator L is known, then by Lemma 4.2 spectra of the operators L_1 and L_2 are known. It is obvious that they form the Borg pair in the interval $\left[0, \frac{\pi}{2}\right]$. By the Borg theorem spectra of these two operators uniquely determine the Sturm-Liouville operator on the segment $\left[0, \frac{\pi}{2}\right]$, and due to the formula $q(x) = q(\pi - x)$, on all the segment $[0, \pi]$. Theorem 4.1 is proved.

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ПОТЕНЦИАЛЫ СИММЕТРИЯЛЫ, АЛ ШЕКАРАЛЫҚ ШАРТТАРЫ АЖЫРАМАЙТЫН ШТУРМ-ЛИУВИЛЛ ОПЕРАТОРЫНЫҢ КЕРІ ЕСЕБІ ТУРАЛЫ

Аннотация. Бұл еңбекте потенциалы симметриялы, нақты әрі үзіксіз, ал шекаралық шарттары ажырамайтын Штурм-Лиувилл операторын бір спектр арқылы анықтауға болатыны көрсетілді. Зерттеу әдісі бұрынғы әдістердің ешбіріне ұқсамайды және ол оператордың ішкі симметриясына негізделген, ал ол өз кезегінде инвариантты кеңістіктердің салдары.

Түйін сөздер: Штурм-Лиувиллдің операторы, спектр, Штурм-Лиувиллдің кері есебі, Боргтың теоремасы, Амбарцумянның теоремасы, Левинсонның теоремасы, ажырамайтын шекаралық шарттар, симметриялы потенциал, инвариантты кеңістіктер.

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ОБРАТНАЯ ЗАДАЧА ОПЕРАТОРА ШТУРМА-ЛИУВИЛЛЯ С НЕ РАЗДЕЛЕННЫМИ КРАЕВЫМИ УСЛОВИЯМИ И СИММЕТРИЧНЫМ ПОТЕНЦИАЛОМ

Аннотация. В данной работе доказана теорема единственности по одному спектру, для оператора Штурма-Лиувилля с не разделенными краевыми условиями и вещественным непрерывным и симметричным потенциалом. Метод исследования отличается от всех известных методов, и основан на внутреннюю симметрию оператора, порожденного инвариантными подпространствами.

Ключевые слова: Оператор Штурма-Лиувилля, спектр, обратная задача Штурма-Лиувилля, теорема Борга, теорема Амбарцумяна, теорема Левинсона, неразделенные краевые условия, симметричный потенциал, инвариантные подпространства.

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REFERENCES

[1] Ambartsumian V.A. Über eine Frage der Eigenwert-theorie, Zsch.f.Physik, 53(1929), 690-695.
[2] Borg G. Eine Umbehrung der Sturm – Liovillschen Eigenwertaufgabe, Acta Math., 78, №2(1946), 1-96.
[3] Levinson N. The inverse Sturm – Lioville problem, Math.Tidsskr. B., 1949, 25-30.
[4] Levitan B.M. On definition of the Sturm-Liouville operator by two spectra. News of AS USSR, ser. Math., 28 (1964), 63-78.
[5] Levitan B.M. On definition of the Sturm-Liouville operator by one and two spectra. News of AS USSR, ser. Math., 42. №1. 1964.
[6] Barcilon V. Iterative solution of the inverse Sturm-Liouville problem. J. Mathematical Phys., 15 (1974), 287-298.
[7] Zhikov V. V. On inverse Sturm-Liouville problems on a finite segment. Math. USSR--Izv.1 (1967), 923-934.
[8] Gasymov G. M., Levitan B. M., On Sturm-Liouville differential operators with discrete spectra. Amer. Math. Soc. Transl. Series 2, 68 (1968), 21-33.

[9] Gelfand I. M., Levitan B. M. On the determination of a differential equation from its spectral function. Amer. Math. Soe. Transl. Series 2, 1 (1955), 253-304.

[10] Hald O. H. Inverse eigenvalue problems for layered media. Comm. Pure Appl. Math.,30 (1977), 69-94.

[11] Hochstad T.H. The inverse Sturm-Liouville problem. Comm. Pure Appl. Math., 26 (1973), 715-729.

[12] Hochstadt H. Well-posed inverse spectral problems. Proc. Nat. Acad. Sci., 72 (1975), 2496-2497.

[13] Hochstadt. H., On the well-posedness of the inverse Sturm-Liouville problem. J. Differential Equations, 23 (1977), 402-413.

[14] Krein M. G., Solution of the inverse Sturm-Liouville problem. Dokl. Akad. Nauk SSSR (N.S.), 76 (1941), 21-24.

[15] Krein, M. G., On the transfer function of a one-dimensional boundary problem of second order.Dokl. Aka:t. 1Vaulc SSSR (N.S.), 88 (1953), 405-408.

[16] Levitan B. M., On the determination of a Sturm-Liouville equation by two spectra. Amer. Math. Soc. Transl. Series 2, 68 (1968), 1-20.

[17] Levitan B. M., Generalized translation operators, Israel Program for Scientific Translations, Jerusalem,1964.

[18] Marchenko V. A., Concerning the theory of a differential operator of the second order. Dokl. Akad. Nauk SSSR (N.S.), 72 (1950), 457-460.

[19] Marchenko V. A., Expansion into eigenfunctions of non-selfadjoint singular differential operators of second order. Amer. Math. Soc. Transl. Series 2, 25 (1963), 77-130.

[20] Hald O.H. The Inverse Sturm--Liouville Problem with symmetric Potentials, Acta mathematica 1978, 141:1, 263 .

[21] Coddinoton E., Levinson, N., Theory of ordinary differential equations. McGraw-Hill, New York, 1955.

[22] Neumark M. A., Lineare Differential Operatoren. Akademie-Verlag, Berlin, 1963.

[23] Titchmarsh E. C., The theory of functions. Oxford University Press, London, 1939.

[24] Titchmarsh E. C., Eigenfunction expansions associated with second order differential equations. Oxford University Press, London, 1946.

[25] Zygmund A., Trigonometric series, vol. I, 2nd ed. Cambridge University Press, London,

[26] Jörgens K., Spectral theory of second-order ordinary differential operators, Lectmyes delivered at Aarhus Universitet, 1962/63, Aarhus, 1964.

[27] Akylbayev M.I., Beysebayeva A., Shaldanbayev A. Sh., On the Periodic Solution of the Goursat Problem for a Wave Equation of a Special Form with Variable Coefficients. News of the National Academy of Sciences of the republic of Kazakhstan. Volume 1, Number 317 (2018), 34 – 50.

[28] Shaldanbaeva A. A., Akylbayev M.I., Shaldanbaev A. Sh., Beisebaeva A.Zh., The Spectral Decomposition of Cauchy Problem's Solution for Laplace Equation. News of the National Academy of Sciences of the republic of Kazakhstan. Volume 5, Number 321 (2018), 75 – 87. https://doi.org/10.32014/2018.2518-1726.10