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TO THE QUESTION OF A MULTIPOINT MIXED BOUNDARY VALUE PROBLEM FOR A WAVE EQUATION

Abstract. It is well known that some problems in mechanics and physics lead to partial differential equations of the hyperbolic type. A classical example of the hyperbolic type is wave equation. When posed, the task sometimes lacks the classical boundary condition and the need arises to have a nonlocal boundary condition. Aim our work is get D'Alembert formula for mixed boundary value problem generated by a wave equation. In the classical case, given D'Alembert formula for boundary value problem generated by a wave equation. In our case, we must give D'Alembert formula for mixed boundary value problem. For this, we consider ordinary differential operator $\mathcal{L}$ withnon–local boundary conditions. We search the solution of the wave equation like a sum with eigenfunction of the operator $\mathcal{L}$. There are we use that fact, that eigenfunction of the operator $\mathcal{L}$ is Riesz basis in $L^2(0,l)$. Through this method and calculation we get D'Alembert formula.

Key words: D'Alembert formula, wave equation, mixed boundary value problem, nonlocal boundary condition.

1 Introduction

It is well known that some problems in mechanics and physics lead to partial differential equations of the hyperbolic type. When posed, the task sometimes lacks the classical boundary condition and the need arises to have a nonlocal boundary condition (see, [2, 3, 4, 5]). A simple example of such nonlocal conditions are multipoint conditions relating the value of the solution at the boundary points with the values at some interior points. For example, we refer the reader to [6, 7, 8, 9, 10].

2 Main result

Let us consider mixed boundary value problem generated by the homogeneous wave equation

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = 0, (x,t) \in \mathrm{S} \tag{1}$$

with inhomogeneous initial data

$$U(x,0) = f(x), U_t(x,0) = F(x), x \in [0,l], \tag{2}$$

and with non–local conditions

$$U(0,t) = 0, \sum_{j=0}^{N} \alpha_j U_x{}'(x_j,t) = 0, \tag{3}$$

where $\mathrm{S} = \{(x,t): 0 < x < l, t > 0\}$,

$$0 = x_0 < x_1 <\ldots< x_N = l, \alpha_0 \neq 0, \alpha_N \neq 0, \sum_{j=0}^{N} \alpha_j = 1,\ l < \infty,$$

the system of point $\{x_j\}_{j=0}^{N}$ on the segment $[0, l]$ is chosen such that the relation $\frac{x_j}{x_{j+1}}$ is a rational number for all $j \geq 0$.

Also consider an ordinary differential operator $\mathcal{L}$ with the expression

$$\mathcal{L}(y) \equiv -y''(x), 0 < x < b, \tag{4}$$

with non–local boundary conditions

$$y(0) = 0, \sum_{j=0}^{N} \alpha_j y'(x_j) = 0. \tag{5}$$

By $\{\lambda_k\}_{k=1}^{\infty}$ denote eigenvalues of $\mathcal{L}$, which are zeros of the characteristic function

$$\Phi(\lambda) = \sum_{j=0}^{N} \alpha_j cos\sqrt{\lambda}x_j.$$

Theorem 1 *A solution of the boundary value problem (1)–(3) has the form*

$$U(x,t) = \frac{\tilde{f}(x+t) + \tilde{f}(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \tilde{F}(\tau)d\tau.$$

Proof. The system of eigen- and associated functions has the form

$$\{X_{kn}(x) = \frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}(\frac{sin\sqrt{\lambda}x}{\sqrt{\lambda}})|_{\lambda=\lambda_n}, k = 0,1,\ldots,m_n - 1\}_{n=1}^{\infty}, \tag{6}$$

which is the Riesz basis in $L^2(0, l)$ (see [1]). Here m_n is a multiplicity of the corresponding eigenvalue λ_n for all $n \in \mathbb{N}$.

Using this fact, let us prove solvability of the problem (1)-(3). The solution of the problem (1)-(3) we will seek in the view

$$U(x,t) = \sum_{n=1}^{\infty} \sum_{k=0}^{m_n-1} d_{kn}(t)X_{kn}(x), \tag{7}$$

where d_{kn} is a coefficient of the Fourier decomposition of U, depends on second argument.

By differentiating twice respect to x (7), we get

$$\frac{\partial^2 U}{\partial x^2} = \sum_{n=1}^{\infty} \sum_{k=0}^{m_n-1} d_{kn}(t)\left[-\lambda_n X_{kn}(x) - X_{k-1,n}(x)\right] =$$

$$\sum_{n=1}^{\infty} \sum_{k=0}^{m_n-1} [-\lambda_n d_{kn}(t) - d_{k+1,n}(t)]X_{kn}(x), \tag{8}$$

where $d_{m_n,n} = 0$.

By differentiating twice respect to t (7), we have

$$\frac{\partial^2 U}{\partial t^2} = \sum_{n=1}^{\infty} \sum_{k=0}^{m_n-1} d_{kn}''(t)X_{kn}(x), \tag{9}$$

By using the condition $U(0, t) = 0$, from (8)-(9) we take

$$d_{kn}''(t) = -\lambda_n d_{kn}(t) - d_{k+1,n}(t), k < m, where d_{m_n,n} = 0. \tag{10}$$

Use initial conditions (2):

$$U(x,0)=\sum_{n=1}^{\infty}\sum_{k=0}^{m_n-1} d_{kn}(0)X_{kn}(x)=f(x),$$

$$U_t(x,0)=\sum_{n=1}^{\infty}\sum_{k=0}^{m_n-1} d_{kn}'(0)X_{kn}(x)=F(x).$$

We calculate Fourier coefficients using the biorthogonal system of $\{X_{kn}(x)\}$ by formulas

$$\begin{aligned} d_{kn}(0)=d_{kn}^{f}=\int_0^b f(x)\overline{h_{kn}(x)}dx,\\ d_{kn}'(0)=d_{kn}^{F}=\int_0^b F(x)\overline{h_{kn}(x)}dx. \end{aligned} \tag{11}$$

Indeed, (10) – (11) is a Cauchy problem, and it's solution has the form

$$d_{kn}(t)=d_{kn}^{f}cos\sqrt{\lambda_n}t+d_{kn}^{F}\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}-\int_0^t\frac{sin\sqrt{\lambda_n}(t-\xi)}{\sqrt{\lambda_n}}d_{k+1,n}d\xi. \tag{12}$$

Taking into account $d_{m_n,n}=0$, for $k=m_n-1$ we have

$$d_{m_n-1,n}(t)=d_{m_n-1,n}^{f}cos\sqrt{\lambda_n}t+d_{m_n-1,n}^{F}\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}.$$

And for $k=m_n-2$ we have

$$d_{m_n-2,n}(t)=d_{m_n-2,n}^{f}cos\sqrt{\lambda_n}t+$$

$$d_{m_n-2,n}^{F}\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}-\int_0^t\frac{sin\sqrt{\lambda}(t-\zeta)}{\sqrt{\lambda}}d_{m_n-1,n}(t)d\zeta$$

$$=d_{m_n-2,n}^{f}cos\sqrt{\lambda_n}t-\int_0^t\frac{sin\sqrt{\lambda}(t-\zeta)}{\sqrt{\lambda_n}}d_{m_n-1,n}^{f}cos\sqrt{\lambda_n}td\zeta$$

$$+d_{m_n-2,n}^{F}\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}-\int_0^t\frac{sin\sqrt{\lambda}(t-\zeta)}{\sqrt{\lambda_n}}d_{m_n-1,n}^{F}\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}d\zeta.$$

We denote

$$C_{0n}(t)=cos\sqrt{\lambda_n}t, S_{0n}(t)=\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}},$$

$$C_{kn}(t)=-\int_0^t\frac{sin\sqrt{\lambda_n}(t-\zeta)}{\sqrt{\lambda_n}}cos\sqrt{\lambda_n}\zeta d\zeta,$$

$$S_{kn}(t)=-\int_0^t\frac{sin\sqrt{\lambda_n}(t-\zeta)}{\sqrt{\lambda_n}}\frac{sin\sqrt{\lambda_n}\zeta}{\sqrt{\lambda_n}}d\zeta,$$

and

$$\begin{aligned} C_{k+1,n}(t)=\frac{1}{k!}\frac{\partial^{k+1}}{\partial\lambda^{k+1}}C_{0n}(t),\\ S_{k+1,n}(t)=\frac{1}{k!}\frac{\partial^{k+1}}{\partial\lambda^{k+1}}S_{0n}(t). \end{aligned} \tag{13}$$

Therefore

$$d_{m_n-2,n}(t) = d^f_{m_n-2,n} cos\sqrt{\lambda_n} t + d^F_{m_n-2,n} \frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}} -$$

$$d^f_{m_n-1,n} C_{1,n}(t) - d^F_{m_n-1,n} S_{1,n}(t).$$

Analogically for another k the function $d_{kn}(t)$ can be written as

$$d_{kn}(t) = \sum_{j=k}^{m_n-1} [d^f_{jn} C_{j-k,n}(t) + d^F_{jn} S_{j-k,n}(t)]. \tag{14}$$

By substituting found $d_{kn}(t)$ in (7), we get

$$U(x,t) = \sum_{n=1}^{\infty} \sum_{k=0}^{m_n-1} \sum_{j=k}^{m_n-1} [d^f_{jn} C_{j-k,n}(t) + d^F_{jn} S_{j-k,n}(t)] X_{kn}(x)$$

$$= \sum_{n=1}^{\infty} \sum_{j=0}^{m_n-1} \{d^f_{jn} \sum_{k=0}^{j} C_{j-k,n}(t) X_{kn}(x) + d^F_{jn} \sum_{k=0}^{j} S_{j-k,n}(t) X_{kn}(x)\}.$$

By virtue of (6) and (13), we have a new presentation

$$U(x,t) = \sum_{n=1}^{\infty} \sum_{j=0}^{m_n-1} \{d^f_{jn} \sum_{k=0}^{j} \frac{1}{(j-k)!} \frac{\partial^{j-k}}{\partial \lambda_n^{j-k}} (cos\sqrt{\lambda_n} t) \frac{1}{k!} \frac{\partial^k}{\partial \lambda_n^k} (\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}})$$

$$+ d^F_{jn} \sum_{k=0}^{j} \frac{1}{(j-k)!} \frac{\partial^{j-k}}{\partial \lambda_n^{j-k}} (\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}) \frac{1}{k!} \frac{\partial^k}{\partial \lambda_n^k} (\frac{sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}})\}.$$

Syne $\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}} = \int_0^t cos\sqrt{\lambda_n}\tau d\tau$, then

$$U(x,t) = \sum_{n=1}^{\infty} \sum_{j=0}^{m_n-1} d^f_{jn} \frac{1}{j!} \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} \frac{\partial^{j-k}}{\partial \lambda_n^{j-k}} (cos\sqrt{\lambda_n} t) \frac{\partial^k}{\partial \lambda_n^k} (\frac{sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}})$$

$$+ \int_0^t d\tau \sum_{n=1}^{\infty} \sum_{j=0}^{m_n-1} d^F_{jn} \frac{1}{j!} \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} \frac{\partial^{j-k}}{\partial \lambda_n^{j-k}} (cos\sqrt{\lambda_n} \tau) \frac{\partial^k}{\partial \lambda_n^k} (\frac{sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}).$$

And using $\sum_{k=0}^{j} C_k^j U^{(k)}(x) V^{(j-k)}(x) = (UV)^{(j)}$, we take

$$U(x,t) = \sum_{n=1}^{\infty} \sum_{j=0}^{m_n-1} d^f_{jn} \frac{1}{j!} \frac{\partial^j}{\partial \lambda_n^j} (cos\sqrt{\lambda_n} t \frac{sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}})$$

$$+ \int_0^t d\tau \sum_{n=1}^{\infty} \sum_{j=0}^{m_n-1} d^F_{jn} \frac{1}{j!} \frac{\partial^j}{\partial \lambda_n^j} (cos\sqrt{\lambda_n} \tau \frac{sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}})$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{f} \frac{1}{j!}\frac{\partial^j}{\partial\lambda_n^j}\left(\frac{\sin\sqrt{\lambda_n}(x+t)}{\sqrt{\lambda_n}}\right)$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{f} \frac{1}{j!}\frac{\partial^j}{\partial\lambda_n^j}\left(\frac{\sin\sqrt{\lambda_n}(x-t)}{\sqrt{\lambda_n}}\right)$$

$$+\frac{1}{2}\int_0^t d\tau \sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{F} \frac{1}{j!}\frac{\partial^j}{\partial\lambda_n^j}\left(\frac{\sin\sqrt{\lambda_n}(x+\tau)}{\sqrt{\lambda_n}}\right)$$

$$+\frac{1}{2}\int_0^t d\tau \sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{F} \frac{1}{j!}\frac{\partial^j}{\partial\lambda_n^j}\left(\frac{\sin\sqrt{\lambda_n}(x-\tau)}{\sqrt{\lambda_n}}\right).$$

So,

$$\begin{aligned} U(x,t) &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{f} X_{jn}(x+t) + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{f} X_{jn}(x-t) \\ &+ \frac{1}{2}\int_x^{x+t}\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{F} X_{jn}(\tau)d\tau + \frac{1}{2}\int_{x-t}^{x}\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{F} X_{jn}(\tau)d\tau. \end{aligned} \quad (15)$$

When $0 \le x - t \le x + t \le b$, then sums of series are coincides with the initial data

$$\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{f} X_{jn}(x+t) = f(x+t),$$

$$\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{f} X_{jn}(x-t) = f(x-t),$$

$$\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{F} X_{jn}(\tau) = F(\tau).$$

For $0 \le x - t \le x + t \le b$ the solution is well–known D'Alembert formula

$$U(x,t) = \frac{f(x+t)+f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} F(\tau)d\tau.$$

Thus, the formula (15) can be interpreted as a generalization of the D'Alembert formula for arbitrary $0 \le x \le b, t \ge 0$.

As a result, we conclude

$$U(x,t) = \frac{\tilde{f}(x+t)+\tilde{f}(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \tilde{F}(\tau)d\tau,$$

where $\tilde{f}(x)$ and $\tilde{F}(x)$ extended from the segment $[0, l]$ to the whole real axis by the analytical continuation of the basis system

$$\{X_{kn}(x) = \frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}\frac{\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}\}.$$

Since $\{X_{kn}(x)\}$ is defined on $(-\infty, +\infty)$, then we can always to continue functions $f(x)$ and $F(x)$ outside of the segment $[0, l]$. From [1] follows that the system $\{X_{kn}(x)\}$ is Riesz basis in $L_2(-\infty, +\infty)$.

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ТОЛҚЫН ТЕҢДЕУІ ҮШІН КӨП НҮКТЕЛІ АРАЛАС ШЕКАРАЛЫҚ ЕСЕП

Аннотация.Механика мен физиканың кейбір есептері дербес туындылы дифференциалдық теңдеулердің гиперболалық түріне алып келетіні белгілі. Гиперболалық теңдеудің классикалық өкіліне толқын теңдеуі жатады. Кейде есеп шығару барысында тек шекаралық шарт жеткіліксіз болады, сондықтан қосымша локалды емес шекаралық шартта қолданылады. Біздің жұмыстың мақсаты толқын теңдеуі арқылы туындаған аралас шекаралық есептің Даламбер формуласын табу. Классикада толқын теңдеуі арқылы туындаған шекаралық есеп үшін Даламбер формуласы берілген. Біз аралас шекаралық есеп үшін Даламбер формуласын табу керекпіз. Ол үшін біз қосымша $\mathcal{L}$дифференциалдық операторын қарастырамыз. Себебі біз шешімді $\mathcal{L}$ операторының меншікті функциялары арқылы құрылған қатар арқылы іздейміз. Біз бұл жерде $\mathcal{L}$ операторының меншікті функциялары $L^2(0,l)$кеңістігінде Рисс базисы болатынын пайдаланамз. Біз осы әдіс және есептеулер арқылы Даламбер формуласын аламыз.

Түйін сөздер:Даламбер формуласы, толқын теңдеуі, аралас шекаралық есеп, локалды емес шекаралық шарт.

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О МНОГОТОЧЕЧНОЙ ЗАДАЧЕ СМЕШАННОЙ ГРАНИЦЫ ДЛЯ ВОЛНОВОГО УРАВНЕНИЯ

Аннотация.Хорошо известно, что некоторые проблемы механики и физики приводят к уравнениям в частных производных гиперболического типа. Классическим примером гиперболического типа является волновое уравнение. При постановке задачи иногда не хватает классического граничного условия, и возникает необходимость иметь нелокальное граничное условие. Цель нашей работы - получить формулу Даламбера для смешанной краевой задачи, порожденной волновым уравнением. В классическом случае дана формула Даламбера для краевой задачи, порожденная волновым уравнением. В нашем случае мы должны дать формулу Даламбера для смешанной краевой задачи. Для этого рассмотрим обыкновенный дифференциальный оператор $\mathcal{L}$ с нелокальными граничными условиями. Мы ищем решение волнового уравнения как сумму с собственной функцией оператора $\mathcal{L}$. Мы используем тот факт, что собственная функция оператора $\mathcal{L}$ является базисом Рисса в $L^2(0,l)$. С помощью этого метода и расчета мы получаем формулу Даламбера.

Ключевые слова:Формула Даламбера, волновое уравнение, смешанная краевая задача, нелокальные краевые условия.

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