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HIERARCY OF WDVV ASSOCIATIVITY EQUATIONS FOR $n=3$ AND $N=2$ CASE WHEN $V_0=0$ WITH NEW SYSTEM a_t, b_t, c_t

Abstract. We investigate solutions of Witten-Dijkgraaf-E.Verlinde-H.Verlinde (WDVV) equations. The article discusses nonlinear equations of the third order for a function f = f(x,t)) of two independent variables x,t. The equations of associativity reduce to the nonlinear equations of the third order for a function f = f(x,t)) when prepotential F dependet of the metric η. In this work we consider the WDVV equation for n = 3 case with an antidiagonal metric η. The solution of some cases of hierarchy equations of associativity illustrated. Lax pairs for the system of three equations, that contains the equation of associativity are written to find the hierarchy of associativity equation. Using the compatibility condition are found the relations between the matrices U, V_2, V_1. The elements of matrix V_2 are found with the expression of z_{ij} and independent and dependent variables for the matrix V_2. Also solving elements of matrix V_1 expressed through y_{ij} and independent and dependent variables for the matrix V_1. We accepted that elements of matrix V_0 are zero. In the physical setting the solutions of WDVV describe moduli space of topological conformal field theories [1, 2]. Let us introduce new variables a, b, c. In the above variables the nonlinear equations of the third order for a function f = f(x,t)) we rewritten as a new system of three equations. Expressed are variables a_t ,b_t ,c_t of three equations are written with the help of matrix elements z_{ij} ,y_{ij}.

Key words: equations of Witten-Dijkgraaf-E.Verlinde-H.Verlinde, the equations of associativity, nonlinear equations of the third order, antidiagonal metric, the Lax pair, the compatibility condition, independent elements, dependent variables, system with equations.

Introduction. The WDVV equations, in general, have the following form [3, 4, 5]:

$$\frac{\partial^3 F}{\partial t^i \partial t^j \partial t^p}\eta^{pq}\frac{\partial^3 F}{\partial t^q \partial t^k \partial t^r}=\frac{\partial^3 F}{\partial t^j \partial t^k \partial t^p}\eta^{pq}\frac{\partial^3 F}{\partial t^i \partial t^q \partial t^r},\quad \forall i,j,k,r\in\{1,...,n\},$$

where F is a prepotential, η is a metric. The coordinates t^i can be linearly rearranged so that the metric, η, is antidiagonal [6], i.e.

$$\eta=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.$$

In this work we consider the WDVV equation for $n=3$ case with an antidiagonal metric η [7]. In this case, two types of dependence of the function F on the fixed variable t^1 were found by Dubrovin [8, 9, 10] which are

$$F=\frac{1}{2}(t^1)^2t^3+\frac{1}{2}t^1(t^2)^2+f(t^2,t^3) \tag{1}$$

and

$$F=\frac{1}{6}(t^1)^3+t^1t^2t^3+f(t^2,t^3).$$

For these cases the equations of associativity reduce to the following two nonlinear equations of the third order for a function $f=f(x,t))$ of two independent variables ($x=t^2, t=t^3$):

$$f_{ttt}=f_{xxt}^2-f_{xxx}f_{xtt} \tag{2}$$

and

$$f_{xxx}f_{ttt}-f_{xxt}f_{xtt}=1,$$

correspondingly.

The function F in equation (1) has the form from the law of multiplication in the three-dimensional algebra A_t with the basis $e_1 = 1$, e_2 ,e_3 [3]. Every basis is a complete uniformly minimal system [11].

In this work, we consider the solution (1). Let us introduce new variables a,b,c as follows [12, 13]:

$$a=f_{xxx},\quad b=f_{xxt},\quad c=f_{xtt}.$$

In the above variables the equation (2) can be rewritten as a system of three equations as follows:

$$\begin{aligned} a_t &= b_x, \\ b_t &= c_x, \\ c_t &= (b^2-ac)_x. \end{aligned} \tag{3}$$

The Lax pair for the system (3) is given by [8]

$$\begin{aligned} \Psi_x &= \lambda U\Psi, \\ \Psi_t &= \lambda V\Psi, \end{aligned} \tag{4}$$

where U is given by

$$U=\begin{pmatrix} 0 & 1 & 0 \\ b & a & 1 \\ c & b & 0 \end{pmatrix}$$

and V is given by

$$V=\begin{pmatrix} 0 & 0 & 1 \\ c & b & 0 \\ (b^2-ac) & c & 0 \end{pmatrix}.$$

The compatibility condition for the system (4) is given by

$$\begin{aligned} U_t &= V_x, \\ [U,V] &= 0. \end{aligned}$$

In the following sections we work with the new system (3).

Methods. The solution to a hierarchy for $N=1$ case corresponds to the system of equations (3). Hierarchy for $N=2$ case when $V_0 \neq 0$ is given in the work [14]

In this section we consider a hierarchy for $N=2$ case when $V_0=0$ and the following system

$$\begin{aligned} a_t &= \varepsilon_1 b_x + \varepsilon_2 F_x, \\ b_t &= \varepsilon_1 c_x + \varepsilon_2 H_x, \\ c_t &= \varepsilon_1 (b^2 - ac)_x + \varepsilon_2 G_x. \end{aligned} \tag{5}$$

The Lax representation of the above system is same as before in the work [13].
In particular, for $N=2$ case when $V_0=0$ we have

$$\begin{aligned} \Psi_x &= \lambda U \Psi, \\ \Psi_t &= (\lambda^2 V_2 + \lambda V_1)\Psi = V\Psi \end{aligned}$$

The compatibility condition of (4) is given by

$$\lambda U_t - V_x + \lambda [U, V] = 0.$$

The compatibility condition of the Lax representation is given by the system

$$[U, V_2] = 0, \tag{6}$$

$$U_t = V_{1x}, \tag{7}$$

$$V_{2x} = [U, V_1] \tag{8}$$

Statement of problem. We first consider the second equation of the system and let V_1 to be given by

$$V_1 = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}.$$

From the above system it follows that $y_{11}, y_{12}, y_{13}, y_{23}, y_{33}$ are constants w.r.t. x. Writing a system with equations for a_t, b_t, c_t only yields

$$\begin{aligned} a_t &= y_{22x}, \\ b_t &= y_{21x}, \\ b_t &= y_{32x}, \\ c_t &= y_{31x}. \end{aligned} \tag{9}$$

Now we equate similar terms in the systems (5) and (9), i.e. we have a system

$$\begin{aligned} a_t &= y_{22x} = \varepsilon_1 b_x + \varepsilon_2 F_x, \\ b_t &= y_{21x} = \varepsilon_1 c_x + \varepsilon_2 H_x, \\ b_t &= y_{32x} = \varepsilon_1 c_x + \varepsilon_2 H_x, \\ c_t &= y_{31x} = \varepsilon_1 (b^2 - ac)_x + \varepsilon_2 G_x. \end{aligned} \tag{10}$$

Scheme of the method and reduction to equivalent problem. From the above system (10) we find the following

$$\begin{aligned} y_{22} &= \varepsilon_1 b + \varepsilon_2 F, \\ y_{21} &= \varepsilon_1 c + \varepsilon_2 H, \\ y_{32} &= \varepsilon_1 c + \varepsilon_2 H, \\ y_{31} &= \varepsilon_1 (b^2 - ac) + \varepsilon_2 G. \end{aligned}$$

Thus the matrix V_1 has the form

$$V_1 = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ \varepsilon_1 c + \varepsilon_2 H & \varepsilon_1 b + \varepsilon_2 F & y_{23} \\ \varepsilon_1 (b^2 - ac) + \varepsilon_2 G & \varepsilon_1 c + \varepsilon_2 H & y_{33} \end{pmatrix}. \tag{11}$$

Now we solve the equation (6). Denote V_2 as follows:

$$V_2 = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix},$$

Plugging U, V_2 into (6) we obtain the following relations:

$$\begin{aligned} z_{23} &= z_{12}, \\ z_{32} &= z_{21}, \\ z_{33} &= z_{11}. \end{aligned}$$

Hence, we are left with the equations

$$\begin{aligned} z_{21} &= b z_{12} + c z_{13}, \\ z_{22} &= z_{11} + a z_{12} + b z_{13}, \\ z_{31} &= c z_{12} + (b^2 - ac) z_{13}. \end{aligned}$$

Thus the matrix V_2 has the form

$$V_2 = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ b z_{12} + c z_{13} & z_{11} + a z_{12} + b z_{13} & z_{12} \\ c z_{12} + (b^2 - ac) z_{13} & b z_{12} + c z_{13} & z_{11} \end{pmatrix}.$$

Hence, only z_{11}, z_{12}, z_{13} are independent elements of V_2, and the other elements can be written in terms of them.

Now let us find the elements of V_1 in (11). To do so we use the equation (8). First we evaluate $[U, V_1]$.

We have elementwise yields the following system:

11: $z_{11x} = \varepsilon_1 c + \varepsilon_2 H - by_{12} - cy_{13}$,
12: $z_{12x} = \varepsilon_1 b + \varepsilon_2 F - y_{11} - ay_{12} - by_{13}$,
13: $z_{13x} = y_{23} - y_{12}$,
21: $b_x z_{12} + bz_{12x} + c_x z_{13} + cz_{13x} = by_{11} + a(\varepsilon_1 c + \varepsilon_2 H) + (\varepsilon_1 (b^2 - ac) + \varepsilon_2 G) - b(\varepsilon_1 b + \varepsilon_2 F) - cy_{23}$,
22: $z_{11x} + a_x z_{12} + az_{12x} + b_x z_{13} + bz_{13x} = by_{12} - by_{23}$,
23: $z_{12x} = by_{13} + ay_{23} + y_{33} - (\varepsilon_1 b + \varepsilon_2 F)$,
31: $c_x z_{12} + cz_{12x} + (b^2 - ac)_x z_{13} + (b^2 - ac) z_{13x} = cy_{11} - cy_{33}$,
32: $b_x z_{12} + bz_{12x} + c_x z_{13} + cz_{13x} = cy_{12} + b(\varepsilon_1 b + \varepsilon_2 F) - (\varepsilon_1 (b^2 - ac) + \varepsilon_2 G) - a(\varepsilon_1 c + \varepsilon_2 H) - by_{33}$,
33: $z_{11x} = cy_{13} + by_{23} - (\varepsilon_1 c + \varepsilon_2 H)$.

Now let us express $\varepsilon_1 c + \varepsilon_2 H$, $\varepsilon_1 b + \varepsilon_2 F$, y_{23} in the element 11, 12, 13 of the above system.

$$\begin{aligned} \varepsilon_1 c + \varepsilon_2 H &= z_{11x} + by_{12} + cy_{13}, \\ \varepsilon_1 b + \varepsilon_2 F &= z_{12x} + y_{11} + ay_{12} + by_{13}, \\ y_{23} &= z_{13x} + y_{12}. \end{aligned}$$

Now let us express $\varepsilon_1 (b^2 - ac) + \varepsilon_2 G$ in the element 21 and substitute the values for $\varepsilon_1 c + \varepsilon_2 H$, $\varepsilon_1 b + \varepsilon_2 F$, y_{23}

$$\varepsilon_1 (b^2 - ac) + \varepsilon_2 G = b_x z_{12} + bz_{12x} + c_x z_{13} + 2cz_{13x} - az_{11x} + bz_{12x} + (b^2 - ac) y_{13} + cy_{12}$$

Now let us express y_{33} in the element 23 and substitute the values for $\varepsilon_1 b + \varepsilon_2 F$, y_{23}

$$y_{33} = 2z_{12x} - az_{13x} + y_{11}$$

Hence, dependent elements of V_1 are given by:

$$\begin{aligned} &\varepsilon_1 (b^2 - ac) + \varepsilon_2 G = b_x z_{12} + bz_{12x} + c_x z_{13} + 2cz_{13x} - az_{11x} + bz_{12x} + (b^2 - ac) y_{13} + cy_{12}, \\ &\varepsilon_1 c + \varepsilon_2 H = z_{11x} + by_{12} + cy_{13}, \\ &\varepsilon_1 b + \varepsilon_2 F = z_{12x} + y_{11} + ay_{12} + by_{13}, \\ &y_{23} = z_{13x} + y_{12}, \\ &y_{33} = 2z_{12x} - az_{13x} + y_{11}. \end{aligned} \tag{12}$$

Now let us rewrite the element 22 by substituting the values for y_{23}. So we have

$$z_{11x} + 2bz_{13x} + a_x z_{12} + az_{12x} + b_x z_{13} = 0$$

Now let us rewrite the element 31 by substituting the values for y_{33}. So we have

$$c_x z_{12} + 3cz_{12x} + (b^2 - ac)_x z_{13} + (b^2 - 2ac) z_{13x} = 0$$

Now let us rewrite the element 32 by substituting the values for $\varepsilon_1 b + \varepsilon_2 F$, $\varepsilon_1 (b^2 - ac) + \varepsilon_2 G$, $\varepsilon_1 c + \varepsilon_2 H$, y_{33}. So we have

$$2b_x z_{12} + 4bz_{12x} + 2c_x z_{13} + (3c - ab) z_{13x} = 0$$

Now let us rewrite the element 33 by substituting the values for y_{23}, $\varepsilon_1 c + \varepsilon_2 H$

$$2z_{11x} - bz_{13x} = 0$$

Also, the independent variables z_{11}, z_{12}, z_{13} of the matrix V_2 have to satisfy the following system of equations:

$$\begin{aligned} z_{11x} + 2bz_{13x} + a_x z_{12} + az_{12x} + b_x z_{13} &= 0, \\ c_x z_{12} + 3cz_{12x} + (b^2 - ac)_x z_{13} + (b^2 - 2ac) z_{13x} &= 0, \\ 2b_x z_{12} + 4bz_{12x} + 2c_x z_{13} + (3c - ab) z_{13x} &= 0, \\ 2z_{11x} - bz_{13x} &= 0. \end{aligned} \tag{13}$$

From the above system (13) it follows that

$$z_{13x} = \left(\frac{4a_x b - 2ab_x}{3ac - a^2 b - 10b^2} \right) z_{12} + \left(\frac{4bb_x - 2ac_x}{3ac - a^2 b - 10b^2} \right) z_{13} \tag{14}$$

$$z_{12x} = \left(-\frac{c_x}{3c} - \frac{b^2 - 2ac}{3c} \cdot \frac{4a_x b - 2ab_x}{3ac - a^2 b - 10b^2} \right) z_{12} + \left(-\frac{b^2 - 2ac}{3c} \cdot \frac{4bb_x - 2ac_x}{3ac - a^2 b - 10b^2} - \frac{(b^2 - ac)_x}{3c} \right) z_{13} \tag{15}$$

Results. Using necessary terms in the system (12) in (10), we obtai

$$\begin{aligned} a_t &= \frac{a_x z_{13x}}{2} + a_x y_{12} + b_x y_{13}, \\ b_t &= \frac{b_x z_{13x}}{2} + b_x y_{12} + c_x y_{13}, \\ c_t &= b_{xx} z_{12} + 3b_x z_{12x} + c_{xx} z_{13} + (a_x b + 3c_x - \frac{ab_x}{2}) z_{13x} - a_x z_{11x} + (b^2 - ac)_x y_{13} + c_x y_{12} \end{aligned} \tag{16}$$

We plug $z_{11x}, z_{12x}, z_{13x}$ in (13), (15), (14) into (16) and obtain the following equation

$$\begin{aligned} a_t &= \left(\frac{2ba_x^2 - aa_x b_x}{3ac - a^2 b - 10b^2} \right) z_{12} + \left(\frac{2ba_x b_x - aa_x c_x}{3ac - a^2 b - 10b^2} \right) z_{13} + a_x y_{12} + b_x y_{13}, \\ b_t &= \left(\frac{2ba_x b_x - ab_x^2}{3ac - a^2 b - 10b^2} \right) z_{12} + \left(\frac{2bb_x^2 - ab_x c_x}{3ac - a^2 b - 10b^2} \right) z_{13} + b_x y_{12} + c_x y_{13}, \\ c_t &= \left(b_{xx} - \frac{b_x c_x}{c} + \frac{5abca_x b_x - 4b^3 a_x b_x + 2ab^2 b_x^2 - 3a^2 cb_x^2 + 2b^2 ca_x^2 + 12bca_x c_x - 6acb_x c_x}{3ac^2 - a^2 bc - 10b^2 c} \right) z_{12} \\ &+ \left(c_{xx} - \frac{b_x (b^2 - ac)_x}{c} + \frac{6abcb_x^2 - 4b^3 b_x^2 + 2ab^2 b_x c_x - 3a^2 cb_x c_x + 2b^2 ca_x b_x + 12bcb_x c_x - abca_x c_x - 6acc_x^2}{3ac^2 - a^2 bc - 10b^2 c} \right) z_{13} \\ &+ (b^2 - ac)_x y_{13} + c_x y_{12} \end{aligned} \tag{17}$$

Conclusion. The solution to a hierarchy for $N = 2$ case when system is given by (5) corresponds to the system of equations (17).

So, we considered of some cases of hierarchy of WDVV associativity equations. Lax pairs for the system of three equations, that contained the equation of associativity written to find the hierarchy of associativity equation. Using the compatibility condition are found the relations between the matrices U, V_2, V_1. Thus, we obtained the elements of the matrices V_2, V_1 for case N = 2 when V_0=0 and the above system a_t, b_t, c_t. It was found, that only z_{11} ,z_{12} ,z_{13} are independent elements of V_2, and the other elements can be written in terms of them. From the above system it follows that $y_{11}, y_{12}, y_{13}, y_{23}, y_{33}$

are constants w.r.t. x. It is found, that y_{11}, y_{12}, y_{13} are independent elements of V_1, and the other elements can be written in terms of them and z_{11}, z_{12}, z_{13}. Expressed are variables a_t, b_t, c_t of three equations are written with the help of matrix elements z_{ij}, y_{ij}.

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n = 3 ЖӘНЕ N = 2 ЖАҒДАЙЛАРЫ ҮШІН ЕНГІЗГІЗІЛГЕН ЖАҢА ЖҮЙЕ a_t, b_t, c_t V_0= 0 БОЛҒАНДАҒЫ WDVV АССОЦИАТИВТІЛІК ТЕҢДЕУІНІҢ ИЕРАРХИЯСЫ

Аннотация. Берілген мақалада Виттен – Диджкграф - Е.Верлинде - Г.Верлинде (ВДВВ) теңдеулері зерттеледі. Бұл жұмыста x, t тәуелсіз айнымалыларынан тұратын f = f(x,t) функциясы үшін үшінші ретті сызықты емес теңдеулер талқыланады. Тәуелсіз x, t айнымалыларынан тұратын f = f(x,t) функциясы үшін үшінші ретті сызықты емес теңдеулер F потенциалы η метрикасымен байланысты болғанда келтіріледі. Сонымен қатар ассоциативтілік теңдеулер иерархиясының бірнеше шешімдері сипатталады. Ассоциативтілік теңдеулерінің иерархиясын табу мақсатында ассоциативтілік теңдеулерінен құралған теңдеулер жүйесі үшін Лакс жұптары жазылды. Сәйкестік шартының қолдану арқылы U, V_2, V_1 матрицалары арасындағы қатынастар анықталды. z_{ij} арқылы өрнектелген V_2 матрицасының элементтері мен V_2 матрицасының тәуелді және тәуелсіз айнымалылары есептелінді. y_{ij} арқылы өрнектелген V_1 матрицасының элементтері мен V_1 матрицасының тәуелді және тәуелсіз айнымалылары табылды. Сонымен қатар V_0 матрицасының элементтері нөлге тең деп алынды. Физикалық қолданылуда WDVV ассоциативтілік теңдеуінің шешімі өрістің топологиялық конформдық теориясының модульдерінің кеңістігін сипаттайды. Жаңа айнымалылар енгізілген. Жаңа айнымалыларда f=f(x,t) функциясы үшін үшінші ретті сызықты емес теңдеулер жаңа жүйе арқылы жазылған. Теңдеулер жүйесінен тұратын a_t, b_t, c_t айнымалылары z_{ij}, y_{ij} матрицалық елементтері арқылы өрнектеліп жазылды.

Түйін сөздер: Виттен-Диджграф-Е.Верлинде-Г.Верлинде теңдеулері, ассоциативтілік теңдеуі, үшінші ретті сызықты емес теңдеулер, антидиагональ метрика, Лакс жұптары, үйлесімділік шарты, тәуелсіз элементтер, тәуелді айнымалылар, теңдеулер жүйесі.

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ИЕРАРХИЯ УРАВНЕНИЙ АССОЦИАТИВНОСТИ WDVV ДЛЯ СЛУЧАЯ n = 3 И N = 2 ПРИ V_0= 0 С НОВОЙ СИСТЕМОЙ a_t, b_t, c_t

Аннотация. В данной статье исследуются уравнения Виттена-Диджкграфа-Е.Верлинде-Г.Верлинде (ВДВВ). В работе обсуждаются нелинейные уравнения третьего порядка для функции f = f(x,t) двух независимых переменных x,t. Уравнения ассоциативности сводятся к нелинейным уравнениям третьего порядка для функции f = f(x,t) когда потенциал функии F связан с метрикой η. В этой работе рассматривается уравнение WDVV для случая n = 3 с антидиагональной метрикой η. Описано решение некоторых случаев иерархии уравнений ассоциативности. Для нахождения иерархии уравнений ассоциативности были записаны пары Лакса для системы из трех уравнений, которая содержит уравнения ассоциативности. С применением условия совместности найдены соотношения между матрицами U, V_2, V_1. Были вычислены элементы матрицы V_2, выраженные через z_{ij}, независимые и зависимые переменные матрицы V_2. Также были найдены элементы матрицы V_1, выраженные через y_{ij} , независимые и зависимые переменные матрицы V_1. Элементы матрицы V_0 равны 0. В физическом приложении решение уравнения ассоциативности WDVV описывает пространство модулей топологических конформных теорий поля. Введены новые переменные a, b, c. В новых переменных нелинейные уравнения третьего порядка для функции f = f(x,t) записаны через новую систему трёх уравнений. Выраженные переменные a_t, b_t, c_t системы из трех уравнений были записаны через матричные элементы z_{ij}, y_{ij}.

Ключевые слова: уравнения Виттена-Дижграфа-Е.Верлинде-Г.Верлинде, уравнения ассоциативности, нелинейные уравнения третьего порядка, антидиагональная метрика, пары Лакса, условие совместности, независимые элементы, зависимые переменные, система с уравнениями.

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