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PHASE PORTRAITS OF THE HENON-HEILES POTENTIAL

Abstract. In this paper the Henon-Heiles potential is considered. In the second half of the 20th century, in astronomy the model of motion of stars in a cylindrically symmetric and time-independent potential was studied. Due to the symmetry of the potential, the three-dimensional problem reduces to a two-dimensional problem; nevertheless, finding the second integral of the obtained system in the analytical form turns out to be an unsolvable problem even for relatively simple polynomial potentials. In order to prove the existence of an unknown integral, the scientists Henon and Heiles carried out an analysis of research for trajectories in which the method of numerical integration of the equations of motion is used. The authors proposed the Hamiltonian of the system, which is fairly simple, which makes it easy to calculate trajectories, and is also complex enough that the resulting trajectories are far from trivial. At low energies, the Henon-Heiles system looks integrable, since independently of the initial conditions, the trajectories obtained with the help of numerical integration lie on two-dimensional surfaces, i.e. as if there existed a second independent integral. Equipotential curves, the momentum and coordinate dependences on time, and also the Poincaré section were obtained for this system. At the same time, with the increase in energy, many of these surfaces decay, which indicates the absence of the second integral. It is assumed that the obtained numerical results will serve as a basis for comparison with analytical solutions.

Keywords: Henon-Heiles model, Poincaré section, numerical solutions.

Introduction. Interest in the existence of the third integral of motion for stars moving in the potential of the galaxy revived in the late 50's and early 60's of the last century. Initially it was assumed that the potential has a symmetry and does not depend on time, therefore in cylindrical coordinates (r, θ, z) this will be only a function of r and z. There must be five integrals of motion that are constant for the six-dimensional phase space. However, the integrals can be either isolating or non-isolating. Non-isolating integrals usually fill all available phase spaces and do not restrict the orbit.

By the time Henon and Heiles wrote their pioneer article, there were only two known integrals of motion: total orbital energy and angular momentum per unit mass of the star. It is easy to show that at least two integrals are not isolated. It was also assumed that the third integral was also not isolated, because no analytical solution has been found so far. Nevertheless, observations of stars near the Sun, as well as numerical calculations of the orbits, behaved in some cases as if they obeyed the three isolating integrals of motion.

Henon and Heiles tried to find out if they could find any real proof that there must be a third isolating integral of the motion. Making numerical calculations, they did not complicate the astronomical meaning of the problem; they only demanded that the potential investigated by them be axially symmetric. The authors also suggested that the motion was tied to a plane and passed into the Cartesian phase space (x, y, $\dot{x}$, $\dot{y}$). After some tests they managed to find a real potential. This potential is analytically simple, so that the orbits can be calculated quite easily, but it is still quite complex, so that the types of orbits are nontrivial. This potential is now known as the potential of Henon and Heiles [1-3].

Methods and calculations. The Henon-Heiles potential is undoubtedly one of the simplest, classical

and characteristic examples of open Hamiltonian systems with two degrees of freedom. The above topic was devoted to a large number of research scientists [4-25].

The potential of the Henon-Heiles system is determined by the formula:

$$U(x,y)=\frac{1}{2}(x^2+y^2+2x^2y-\frac{2}{3}y^3) \tag{1}$$

Equation (1) shows that the potential actually consists of two harmonic oscillators, which were connected by the perturbing terms $x^2y-\frac{1}{3}y^3$.

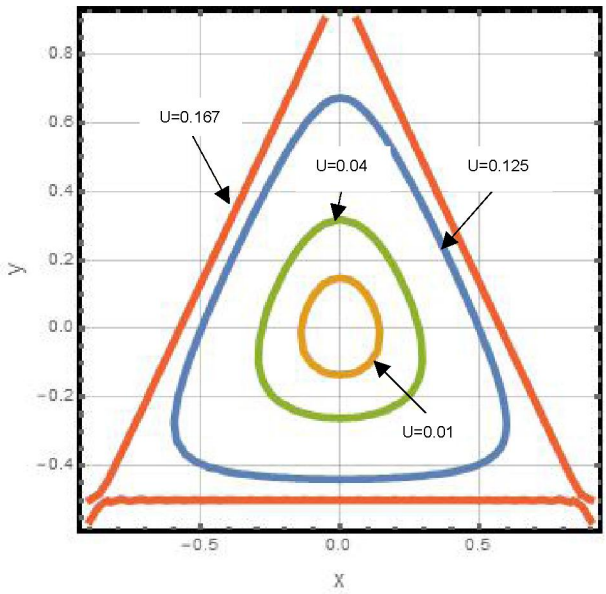

Figure 1 - Closed equipotential curves for the Henon-Heiles model for different values of U

The basic equations of motion for a test particle with a unit mass $(m=1)$ are:

$$\begin{cases} \ddot{x}=-\dfrac{\partial U}{\partial x}=-x-2xy \\ \ddot{y}=-\dfrac{\partial U}{\partial y}=-y-x^2+y^2 \end{cases} \tag{2}$$

Consequently, the Hamiltonian of system (1) has the form:

$$H=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)+\frac{1}{2}\left(x^2+y^2\right)+x^2y-\frac{1}{3}y^3=h, \tag{3}$$

where $\dot{x}$ and $\dot{y}$ are the momenta per unit mass, x and y are the coordinates of the system; $h>0$ the numerical value of the Hamiltonian, which is conserved. It is seen that $h>0$ the Hamiltonian is symmetric with respect to $x\to -x$, and H also exhibits a symmetry of rotation at $2\pi\ /\ 3$.

Below are the dependencies of the coordinates of the functions in time for the systems of equations (2).

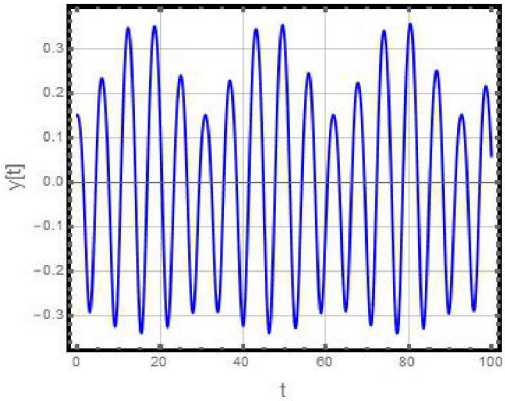

Figure 2 - Dependence of the function y on time

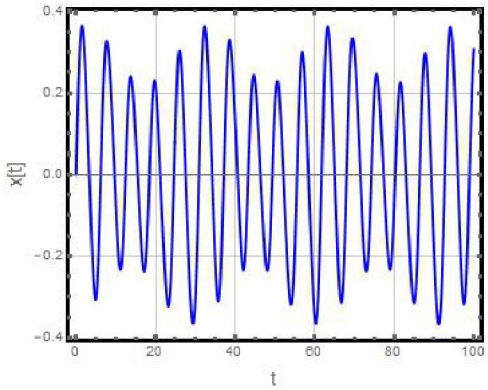

Figure 3 - Dependence of the function x on time

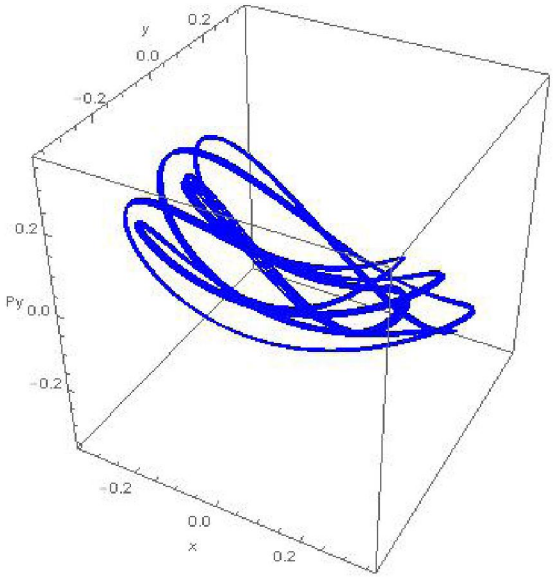

Figure 4 - Evolutionary trajectories of the functions x, y, Py

To study the Henon-Heiles system, the Poincaré section method is used. Advantages of this method are especially evident when we consider nonlinear systems for which exact solutions are unknown. In this case, the phase trajectories are calculated by numerical methods.

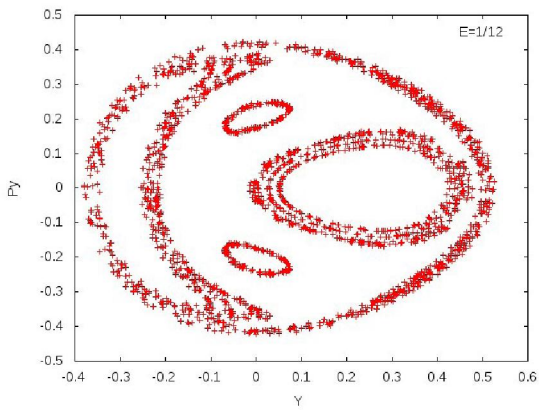

Figure 5 - Poincare section at $E = 1/12$.

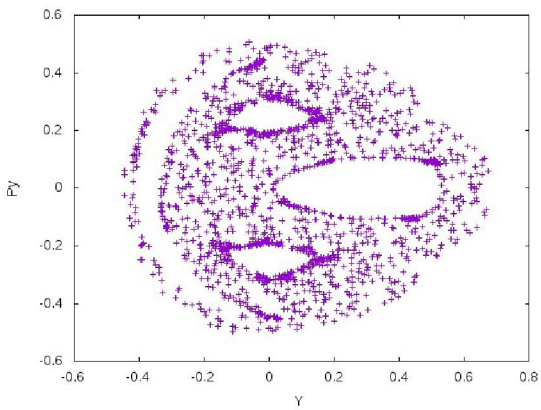

Figure 6 - Poincare section at $E = 1/8$.

To solve the systems of equations (2), boundary conditions are chosen so that they satisfy equation (3). Further, the systems of equation (2) are solved on the basis of the Runge-Kutta method. To construct the Poincaré section, those values that intersect the plane $x = 0$ are chosen. Below are the Poincaré sections for Henon-Heiles systems for different energy values: $E = 1/12$, $E = 1/8$, $E = 1/6$. With increasing energy, the structure of the cross sections is destroyed. The results obtained are in agreement with other authors [1, 2].

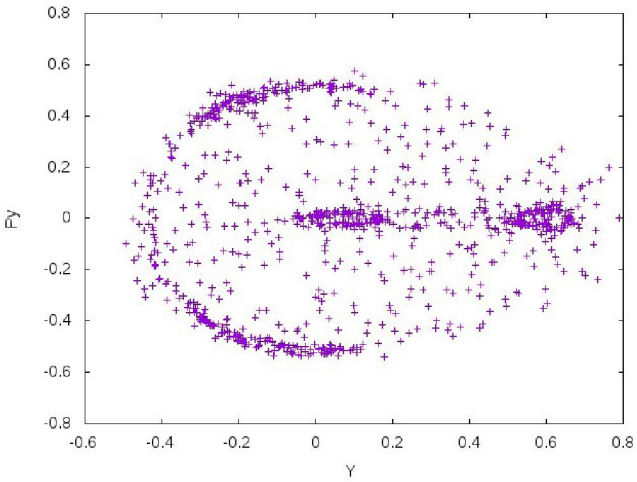

Figure 8 - Poincare section at $E = 1/6$.

Conclusion. Thus, the results obtained by the numerical method determine the oscillations for the Henon-Heiles model and serve as the basis for a comparative analysis in determining the analytical mapping.

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ФАЗОВЫЕ ПОРТРЕТЫ ПОТЕНЦИАЛА ХЕНОНА-ХЕЙЛЕСА

Аннотация. В данной работе исследуется потенциал Хенона-Хейлеса. Во второй половине XX века в астрономии изучались модели движения звезд в цилиндрически симметричном и не зависящем от времени потенциале. Из-за симметрии потенциала трехмерная задача сводится к двумерной, тем не менеенахождение второго интеграла полученной системы в аналитическом виде оказывается неразрешимой задачей даже для сравнительно простых полиномиальных потенциалов. Чтобы доказать существование неизвестного интеграла, ученые Хенон и Хейлес провели анализ исследований для траекторий, в котором используютметод численного интегрирования уравнений движения. Авторы предложили гамильтониан системы, который достаточно прост, что позволяет легко вычислять траектории, а также достаточно сложен, чтобы полученные траектории оказались далеко не тривиальными. При малых энергиях система Хенона-Хейлеса выглядит интегрируемой,так как независимо от начальных условий, траектории, полученные с помощью численного интегрирования, лежат на двумерных поверхностях, т.е. так, как если бы существовал второй независимый интеграл. Для данной системы были получены эквипотенциальные кривые, зависимости импульса и координаты от времени, также сечение Пуанкаре. В то же время с увеличением энергии многие из этих поверхностей распадаются, что указывает на отсутствие второго интеграла. Предполагается что, полученные численные результаты, послужат основой для сравнения с аналитическими решениями.

Ключевые слова: Модель Хенона-Хейлеса, сечение Пуанкаре, численные решения.

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ХЕНОН-ХЕЙЛЕС ПОТЕНЦИАЛЫНЫҢ ФАЗАЛЫҚ БЕЙНЕСІ

Аннотация.Берілген мақалада Хенон-Хейлес моделы қарастырылады. 20 ғасырдың екінші жартысында жұлдыздар қозғалысының цилиндрлік симметриялы және уақыттан тәуелсіз потенциал моделы зерттелді. Потенциалдың симметриялығы арқасында үш өлшемді есепті екі өлшемді есеп ретінде қарастыруға болады, алайда берілген жүйенің аналитикалық түрде екінші интегралын табу, тіпті салыстырмалы түрде қарапайым полиномиальды потенциалдар үшін шешілмейтін мәселе болып табылады. Белгісіз интегралдың бар екенін дәлелдеу үшін, Хенон мен Хейлес қозғалыс теңдеулерінің сандық интегралдау әдісін пайдаланып траектория бойынша зерттеулер жасады. Авторлар жүйенің гамильтонианын ұсынды, ол өте қарапайым және траекторияны есептеуді жеңілдетеді, сонымен қатар алынған траекториялар тривиалды емес күрделі болып табылады. Энергияның төмен деңгейлерінде Хенон-Хейлес жүйесі интегралданады, екінші белгісіз интергралы бар секілді бастапқы шарттардан тәуелсіз жүйенің траекториясы сандық интегралдық әдіспен шешіліп, екі өлшемді кеңістікте сипатталады. Берілген жүйе үшін эквипотенциалдық қисықтар, импульс пен координаттың уақыттан тәуелділігі, сондай-ақ Пуанкаре қимасы алынды. Сонымен қатар, энергияның өсуімен, осы беттердің көпшілігі ыдырайды, бұл екінші интегралдың болмауын көрсетеді. Алынған сандық нәтижелер аналитикалық шешімдермен салыстыру үшін негіз болады деп болжануда.

Түйін сөздер: Хенон-Хейлес моделы, Пуанкаре қимасы, сандық шешімдер.

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