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THE SPECTRAL DECOMPOSITION OF CAUCHY PROBLEM'S SOLUTION FOR LAPLACE EQUATION

Abstract: The spectral decomposition of Cauchy problem for Laplace equation is obtained in Krein space, and is made a regularization of a problem, using the resolvent of the corresponding operator.
Keywords: spectrum, spectral decomposition, equation with deviating argument, Hilbert-Schmidt theorem, Cauchy Problem, Laplaceequation, incorrect, r ange.

1. Introduction.

Currently, there are different approaches to the solution of Cauchy problem for elliptic equations, which is a classic example of an ill-posed problem. All approaches basically can be divided on two large groups. One group consists of methods based on the introduction of the problem into the class of correctness by Tikhonov [1] - [3], the other are the methods using the universal regularizing algorithms, obtained by means of the parametric functional of Tikhonov [4].

It should be noted that the second group of methods received the most spread and major achievements in the practical application. In this approach, are used different variants of regularized algorithms that reduce the problem or to the solution of integral equations of the first kind, or to the representation of the desired field in the region beside or to the construction of finite-difference regularized algorithms [4] — [6].

In view of a great importance of the problem, which has applications in many fields of science and technology, and constantly rising requirements for the reliability of the results, the search for other approaches to its solution is continued. Iterative methods in recent years are becoming more widely used in the practice of solutions of various ill-posed problems of mathematical physics [7] - [9]. These methods have a number of undoubted advantages, which include simple computational schemes, their uniformity for applications with linear and nonlinear operators, the high accuracy of the solution, and so on.

An important advantage is the fact that they allow simple accounting of the essential restrictions for tasks on the solution directly in the scheme of the iterative algorithm (e.g., restrictions on non-negativity of solutions, monotonicity, and so on). In [10] was proposed a new method for solving the problem in question, based on the alternating iterative procedure, which is a consistent solution of the correct mixed boundary value problems for the original equation.

It is prooved the convergence of the method and its regularizing properties. This method is general and can be extended to a wide range of similar ill-posed boundary value problems of mathematical physics. The weak point of the method is the requirement for the smoothness of the boundary, which is not always fulfilled, in particular, in our case. In this paper we propose a spectral method [11-14].

Let $\Omega = [-1,1] \times [0, \pi]$ be a rectangle with sides
$AB: y = 0, -1 \le x \le 1; BC: x = 1, 0 \le y \le \pi; CD: y = \pi, -1 \le x \le 1; DA: x = -1, 0 \le y \le \pi$
(see fig.1)

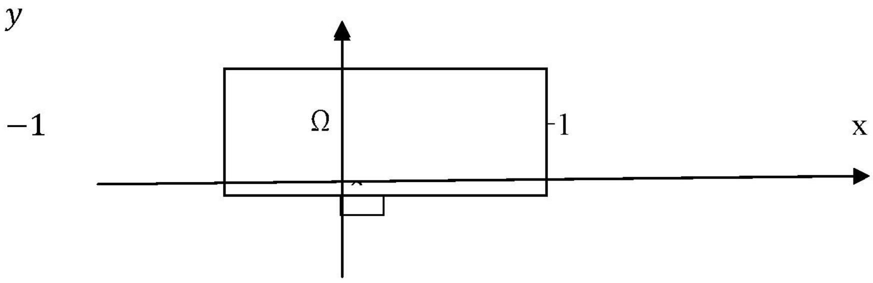

Fig 1.

Let’s consider the following Cauchy - Dirichlet problem for Poisson's equation in the region Ω:

$$Lu = u_{xx} - u_{yy} = f(x, y), \tag{1}$$

$$u|_{y=0} = 0, \quad u|_{y=\pi} = 0, \tag{2}$$

$$u|_{x=-1} = 0, \ \frac{\partial u}{\partial x}\Big|_{x=-1} = 0, \tag{3}$$

where$f(x, y) \in L^2(\Omega)$. This problem has been investigated previously in [11], [12] and is found that inverse operator L^{-1} exists, but is unlimited, in particular, it was shown that the "smallest" eigenvalue of the operator $A = SL$has asymptotics

$$\lambda_{m0} = 4m^2 e^{-2m}[1 + 0(1)], \quad m \to \infty,$$

where the operator S has the form $Su(x, y) = u(-x, y)$.

This work complements and refines the results of these studies.

2. Research Methods

The main idea of the method belongs to T.Sh. Kalmenov [11], and consists in the following. The operator $A = SL$ is symmetric in the space $L^2(\Omega)$, so with the original problem the boundary value problem is studied

$$SLu = Sf, \tag{1'}$$

$$u|_{y=0} = 0, \quad u|_{y=\pi} = 0, \tag{2'}$$

$$u|_{x=-1}, \ \frac{\partial u}{\partial x}\Big|_{x=-1} = 0, \tag{3'}$$

where the operator S has the form, see [13] - [15].

$$Su(x, y) = u(-x, y),$$

and resembles an involution of M.G. Krein, see [16].

The following spectral problem corresponds to this boundary value problem (1 ') - (3')

$Au = \lambda u,$

$$u|_{y=0} = 0, \quad u|_{y=\pi} = 0,$$

$$u|_{x=-1}, \ \frac{\partial u}{\partial x}\Big|_{x=-1} = 0,$$

where

$$A = SL,$$

or in expanded form

$$u_{xx} + u_{yy} = \lambda u(-x, y), \tag{4}$$

$$u|_{y=0} = 0, \quad u|_{y=\pi} = 0,$$

$$u|_{x=-1}, \ \left.\frac{\partial u}{\partial x}\right|_{x=-1} = 0, \tag{5}$$

We solve this spectral problem (4) - (5) using the method of separation of variables, assuming

$$u(x, y) = v(x)w(y),$$

and as a result we get two spectral problems

a) $-w''(y) = \mu w(y),$ (6)

$w(0) = 0, \ w(\pi) = 0;$ (7)

$b)\ v''(x) - \mu v(x) = \lambda v(-x),$ (8)

$v(-1) = 0, v'(-1) = 0.$ (9)

The solution of (6) - (7) is well known and has the form $w_m(y) = \sin my, \ m = 1,2 \dots$,; an analogue of the spectral problem (8) - (9) was investigated in detail in [14], however, we will give a full and detailed study of this problem in the fourth section of the article. As a result, we have

$$Au_{mn} = \lambda_{mn} u_{mn}, \qquad m = 1,2, \dots; \quad n = 0,1,2, \dots$$

where$\{u_{mn}\}$, $m = 1,2, \dots;$ $n = 0,1,2, \dots$is complete and orthonormal system of functions in the space $L^2(\Omega)$.

Further, from the equation (1') we have
$\bar{A}u = Sf,$

Where $\bar{A}$ is a closure of the operator A in the space $L^2(\Omega)$.

Hence,

$$u = (\bar{A})^{-1}Sf = \sum_{m,n}^{\infty} \langle (\bar{A})^{-1}Sf, u_{mn} \rangle u_{mn} = |(\bar{A})^* = \bar{A}| = \langle Sf, (\bar{A})^{-1}u_{mn} \rangle u_{mn}$$
$$= \left| Au_{mn} = \lambda_{mn}u_{mn}, \Rightarrow \bar{A}u_{mn} = \lambda_{mn}u_{mn}, \qquad A \subset \bar{A}, \Rightarrow (\bar{A})^{-1}u_{mn} = \frac{u_{mn}}{\lambda_{mn}} \right|$$
$$= \sum_{m,n}^{\infty} \frac{(Sf, u_{mn})}{\lambda_{mn}} u_{mn} = \sum_{m,n}^{\infty} \frac{[f, u_{mn}]}{\lambda_{mn}} u_{mn},$$

where$[f, u_{mn}] = (Sf, u_{mn})$ is the inner product of Krein's space , and $(.,.)$ is the usual inner product of the space $L^2(\Omega)$, i.e.,

$$(f, g) = \iint_{\Omega} f(x, y) \cdot \overline{g(x, y)} dxdy.$$

Therefore, we need to show a closability of operator A, an essential self-adjointness: $\bar{A} = A^*$, and reversibility: $\ker \bar{A} = \{0\}$, because all of these properties are used in derivation of the last formula

$$u(x,y) = \sum_{m,n}^{\infty} \frac{[f, u_{mn}]}{\lambda_{mn}} u_{mn}(x,y).$$

In addition, it is necessary to examine the spectrum of the operator $\bar{A}$.

3. Results of research.

Let $D(A)$ is a domain of definition of operator A, and $R(A)$ is a domain of its values, $kerA$ is a kernel of the operator A, where

$$A = SL$$

$$Lu = u_{xx} + u_{yy}, \qquad Su(x,y) = u(-x,y),$$

$$D(A) = \left\{u(x,y) \in C^2(\Omega) \cap C^1(\bar{\Omega}): u|_{y=0} = 0, \quad u|_{y=\pi} = 0, u|_{x=-1}, \left.\frac{\partial u}{\partial x}\right|_{x=-1} = 0\right\};$$

We denote through $\bar{A}$ the closure of the operator A in the space $L^2(\Omega)$.

The following theorem holds

Theorem 1.

(a) A is closable, i.e. its closure exists;

(b) A is essentially self-adjoint in the space $L^2(\Omega)$, i.e. the equality holds $(\bar{A})^* = \bar{A}$;

(c) A is invertible, i.e. $ker\bar{A} = \{0\}$, but the inverse operator $(\bar{A})^{-1}$ is unlimited, and has the form

$$u(x,y) = (\bar{A})^{-1} Sf(x,y) = \sum_{m,n}^{\infty} \frac{(Sf, u_{mn})}{\lambda_{mn}} u_{mn}(x,y) = \sum_{m,n}^{\infty} \frac{[f, u_{mn}]}{\lambda_{mn}} u_{mn}(x,y),$$

where$\{u_{mn}\}, m = 1,2,\dots; n = 0,1,2,\dots$ are the orthonormal eigenvectors of A, and $\lambda_{mn}$are the corresponding eigenvalues;

d) $\overline{R(A)} = H = L^2(\Omega) \neq R(\bar{A})$;

i.e. the operator equation

$$\bar{A}u = Sf$$

is densely solvable in the space $L^2(\Omega)$, but not everywhere solvable. The following theorem 2 reveals the spectral properties of the operator $\bar{A}$.

Theorem 2. The spectrum of operator $\bar{A}$ consists of four parts

a) The negative part:

$$-m^2 - (n\pi)^2 < \lambda_{mn}^- < -m^2 - \left(n\pi + \frac{\pi}{4}\right)^2, \quad m,n = 1,2,\dots;$$

b) the "zero" part:

$$m^2 e^{-2\sqrt{2}m cos u_0(m^2)} < \lambda^0(m^2) < 2\left(1+\sqrt{2}\right) m^2 e^{-2\sqrt{2}m \sin u_0(m^2)},$$

where

$$u_0(m^2) \geq u_0(1) > u_0^*(1) > 0,$$

and

$$\lim_{m\to\infty} u_0(m^2) = \frac{\pi}{4}; \quad m = 1,2\dots$$

c) the positive part

$m^2 + (n\pi)^2 < \lambda_{mn}^+ < m^2 + (n\pi + \frac{\pi}{2})^2, \quad m,n = 1,2,\dots$

d) the limit part, i.e. point $\lambda = 0$ belongs to the limit spectrum of the operator $\bar{A}$, i.e. the equality holds

$$\overline{\{\lambda_{mn}\}} \ni \{0\}$$

g) theinequalities hold $\lambda_{mn} \neq 0, \ m = 1,2,\dots; \ n = 0,1,2,\dots$

Theorem 3. The boundary value problem

$u_{xx} + u_{yy} = \lambda u(-x,y), \quad x \in (-1,1], \qquad y(x,\pi)$

$$u|_{y=0} = 0, \quad u|_{y=\pi} = 0,$$

$$u|_{x=-1}, \quad \left.\frac{\partial u}{\partial x}\right|_{x=-1} = 0,$$

has a complete and orthogonal system of eigenvectors:

$u_{mn}(x,y) = \sin my * v_{mn}(x), \qquad m = 1,2,\dots; n = 0,1,2,\dots$

$v_{mn}(x) = K_{mn}\left[ch\sqrt{m^2+\lambda_{mn}}sh\sqrt{m^2-\lambda_{mn}}xsh\sqrt{m^2-\lambda_{mn}}ch\sqrt{m^2+\lambda_{mn}}x\right],$

$$m = 1,2,\dots; n = 0,1,2,\dots$$

whereK_{mn} are the normalization coefficients, and$\lambda_{mn}$are the roots of the equation

$$th\sqrt{m^2-\lambda}th\sqrt{m^2+\lambda} = \frac{\sqrt{m^2-\lambda}}{\sqrt{m^2+\lambda}},$$

for each fixed value m = 1,2, ... All the eigenvalues are simple, real and not equal to zero.

4. The proofs and discussion.

4.1. On the solvability.

Lemma 1. If the eigenvectors of a symmetric operator T, corresponding to non-zero eigenvalues, form an orthonormal basis of the Hilbert space H, then

a) this operator is essentially self-adjoint;

b) the operator $\bar{T}$ is reversible;

c) $\overline{R(\bar{T})} = H$;

d) $\overline{R(\bar{T})} = R(\bar{T})$,

if and only if the inequality holds

$$|\lambda_n| \geq \varepsilon > 0, n = 1,2,\dots$$

where$\lambda_n(n = 1,2,\dots)$ are the eigenvalues of the operator T acting in a Hilbert space H.

4.2. On the spectrum of the operator B.

Consider in the space $L^2(-1,1)$ the following spectral problem

$v''(x) - \mu v(x) = \lambda v(-x), x \in (-1,1]$ (8)

$v(-1) = 0, \ v'(-1) = 0$ (9)

where μ is a fixed real quantity, λ is a spectral parameter.

Let $B = S\hat{L}$, where

$$\hat{L}v = v''(x) - \mu v(x), \qquad Su(x) = u(-x),$$

then the spectral problem (8) - (9) takes the form

$$Bv = \lambda v; \ v(-1) = 0, v'(-1) = 0.$$

Notice that

$$D(B) = \{v(x) \in C^2(-1,1) \cap C^1[-1,1]: v(-1) = 0, v'(-1) = 0\}.$$

Obviously $C_0^\infty(-1,1) \subset D(B)$. In addition, the equality holds

$$(Bu,v) = (u,Bv), \qquad \forall u,v \in D(B)$$

If $\lambda = 0$, then $v(x) = 0$ by virtue of the uniqueness of the solution of Cauchy problem, so $kerB = \{0\}$, i.e. the inverse operator B^{-1} exists, which has the following form

$$v(x) = B^{-1}f(x) = \int_{-1}^{x}\frac{sh\sqrt{\mu}(x-t)Sf(t)}{\sqrt{\mu}}dt,$$

for any continuous function $f(x) \in C[-1,1]$. By means of the extension theorem (see [19]., C.154), we will continue this operator on the whole space $L^2(-1,1)$ as a continuous operator

$$\overline{B^{-1}}f(x) = \int_{-1}^{x} \frac{sh\sqrt{\mu}(x-t)Sf(t)}{\sqrt{\mu}}dt, \ \ \forall f(x) \in L^2(-1,1).$$

It is obvious that the operator $\overline{\mathrm{B}^{-1}}$ is completely continuous and self-adjoint in the space $\mathrm{L}^2(-1,1)$. By the formula,

$$\overline{\mathrm{B}^{-1}} = (\overline{\mathrm{B}})^{-1}$$

we have that the operator $\overline{\mathrm{B}}$ is reversible. Operator $(\overline{\mathrm{B}})^{-1}$ is completely continuous and self-adjoint in $L^2(-1,1)$.

By Hilbert-Schmidt Theorem (see [17], p. 226), for any $f(x) \in L^2(-1,1)$ the formula holds

$$(\bar{B})^{-1}f = \sum_{n=1}^{\infty}\langle(\bar{B})^{-1}f, v_n(\mu;x)\rangle v_n(\mu;x) = \sum_{n=1}^{\infty}\langle f, (\bar{B})^{-1}v_n(\mu;x)\rangle v_n(\mu;x)$$
$$= \sum_{n=1}^{\infty}\langle f, \overline{B^{-1}} \ \ v_n(\mu;x)\rangle\, v_n(\mu;x) = \left|\overline{B^{-1}}v_n = B^{-1}v_n = \frac{v_n(\mu;x)}{\lambda_n(\mu)}\right|$$
$$= \sum_{n=1}^{\infty}\langle f, v_n(\mu;x)\rangle \frac{v_n(\mu;x)}{\lambda_n(\mu)},$$

where$\lambda_n(\mu)$ are the eigenvalues of the operator B, and $v_n(\mu;x)$ are the corresponding eigenvectors.

If $\langle f, v_n(\mu;x)\rangle = 0$, for $n = 1,2,...$, then $(\bar{B})^{-1}f = 0$,hence $f = 0$, i.e. the system $\{v_n(\mu;x)\}, n = 1,2,...$ is complete and orthogonal in the space $L^2(-1,1)$. We formulate the obtained results as following lemma.

Lemma 2. If $\mu = \bar{\mu}$, i.e. it is a real value, then

a) the operator $(\bar{B})^{-1}$ is completely continuous and self-adjoint;

b) the spectrum of $\bar{B}$ is discrete, i.e., it has no the condensation points;

c) the normalized eigenvectors of $\bar{B}$ form the orthonormal basis of the space $L^2(-1,1)$.

Let's find the eigenfunctions of the problem (8) - (9). The general solution of equation (8) has the form

$$v(\mu,\lambda;x) = a(\mu,\lambda)sh\sqrt{\mu-\lambda}x + b(\mu,\lambda)ch\sqrt{\mu+\lambda}x, \quad (10)$$

where$a(\mu,\lambda), b(\mu,\lambda)$ are arbitrary constants.

Indeed,

$$v'(\mu,\lambda;x) = a(\mu,\lambda)\sqrt{\mu-\lambda}ch\sqrt{\mu-\lambda}x + b(\mu,\lambda)\sqrt{\mu+\lambda}sh\sqrt{\mu+\lambda}x, \quad (11)$$
$$v''(\mu,\lambda;x) == a(\mu,\lambda)(\mu-\lambda)sh\sqrt{\mu-\lambda}x + b(\mu,\lambda)(\mu+\lambda)ch\sqrt{\mu+\lambda}x$$
$$= \mu\lambda(\mu,\lambda;x)[-a(\mu,\lambda)sh\sqrt{\mu-\lambda}x + b(\mu,\lambda)ch\sqrt{\mu+\lambda}x]\lambda = \mu v(\mu,\lambda;x) + \lambda v(\mu,\lambda;-x),$$
$$\Rightarrow v''(\mu,\lambda;x) - \mu v(\mu,\lambda;x) = \lambda v(\mu,\lambda;-x).$$

Substituting (10) - (11) into the boundary conditions (9), we have

$$\begin{cases} -ash\sqrt{\mu-\lambda} + bch\sqrt{\mu+\lambda} = 0 \\ a\sqrt{\mu-\lambda}ch\sqrt{\mu-\lambda} - b\sqrt{\mu+\lambda}sh\sqrt{\mu+\lambda} = 0 \end{cases} \quad (*)$$

This system of equations has a nontrivial solution only if its determinant $\Delta(\mu,\lambda)$ equals 0, where

$$\Delta(\mu,\lambda) = \begin{vmatrix} -sh\sqrt{\mu-\lambda} & ch\sqrt{\mu+\lambda} \\ \sqrt{\mu-\lambda}ch\sqrt{\mu-\lambda} & -\sqrt{\mu+\lambda}sh\sqrt{\mu+\lambda} \end{vmatrix}.$$

Expanding this determinant, we obtain

$$\Delta(\mu,\lambda) = \sqrt{\mu+\lambda}sh\sqrt{\mu+\lambda}sh\sqrt{\mu-\lambda} - \sqrt{\mu-\lambda}ch\sqrt{\mu-\lambda}ch\sqrt{\mu+\lambda} \quad (12)$$

If $\lambda = 0$, then

$$\Delta(\mu,0) = \sqrt{\mu}sh^2\sqrt{\mu} - \sqrt{\mu}ch^2\sqrt{\mu} = -\sqrt{\mu}[ch^2\sqrt{\mu} - sh^2\sqrt{\mu}] = -\sqrt{\mu},$$

If in addition $\mu = 0$, then from (8) - (9) we have that $v(x) = 0$. Consequently, the value of $\lambda = 0$ is not an eigenvalue of the problem (8) - (9).

If $\lambda = \mu$, then $\Delta(\mu,\mu) = 0$, therefore, the value of $\lambda = \mu$ is probably the eigenvalue of the problem (8) - (9), to which the following eigenfunction corresponds

$$v(\mu,\mu;x) = b(\mu,\mu)ch\sqrt{2\mu}x.$$

But this function satisfies the boundary condition (9) only when the $b(\mu,\mu) = 0$, so there is no eigenvalues of the boundary value problem (8) - (9) in the segment $[-\mu, 0]$, $\mu \geq 0$.

Assuming $\mu > 0$ for definiteness, let us study the distribution of zeros of functions (12).

Lemma 3. If $\mu > \mu_0$, then the function $F(\mu,u)$has a unique simple zero u_0, located in the interval$0 < u_0^* < u_0 < \frac{\pi}{4}$, where

$$1 - \sqrt{2\mu_0}th\sqrt{2\mu_0} = 0, \quad \frac{\partial F}{\partial u}\Big|_{u=u_0^*} = 0.$$

Lemma 4. If $z = u$ a real quantity, the

a) for $0 < \mu < \mu_0$ segment $[-\mu,\mu]$no eigenvalues;

b) when $\mu \geq \mu_0$in interval $(0,\mu)$will be exactly one eigenvalue $\lambda_0^+(\mu)$, which satisfies the estimate

$$\mu \cdot e^{-2\sqrt{2\mu}\cos u_0(\mu)} < \lambda_0^+(\mu) < 2\mu\big(1+\sqrt{2}\big) \cdot e^{-2\sqrt{2\mu}\sin u_0(\mu)},$$

where

$$u_0(\mu) > u_0^*(\mu) > 0, \quad \forall\mu > \mu_0, \qquad \lim_{\mu\to+\infty} u_0(\mu) = \frac{\pi}{4},$$

$$\frac{\partial F}{\partial u}\Big|_{u=u_0^*} = 0, \quad 1 - \sqrt{2\mu_0}th\sqrt{2\mu_0} = 0.$$

a) $0 < \mu \leq \mu_0$

$-\infty + \infty$

$-\mu 0 \qquad + \mu$

б) $\mu > \mu_0 > 0$

$-\infty\lambda_0^+ \qquad + \infty$

$-\mu 0 \qquad + \mu$

fig 5.

Consequence. Forall $\mu \geq 1$, the inequality holds

$$\mu \cdot e^{-2\sqrt{2\mu}} < \lambda_0^+(\mu) < 2\mu(1+\sqrt{2})e^{-2\sqrt{2\mu}\sin u_0^*(1)}$$

where

$$\frac{\partial u}{\partial u}\Big|_{u=u_0^*(1)} = 0, \quad u_0^*(1) > 0.$$

Lemma 5.

a) If $0 < \mu < \mu_0$, then the eigenvalues of the spectral task

$$v''(x) - \mu v(x) = \lambda v(-x), \quad x \in (-1,1] \tag{8}$$

$$v(-1) = 0, \quad v'(-1) = 0 \tag{9}$$

consists of two series: negative

$$-\left[\mu + \left(n\pi + \frac{\pi}{2}\right)^2\right] < \lambda_n^- < -\left[\mu + \left(n\pi + \frac{\pi}{4}\right)^2\right], \quad n = 0,1,2,...,$$

and positive

$$\mu + (n\pi)^2 < \lambda_n^+(\mu) < \mu + (n\pi + \frac{\pi}{2})^2, \quad n = 0,1,2 ...$$

b) If $\mu > \mu_0$, then the third "zero" series will appear, that lays in the interval $(0, \mu)$, for which a two-sided estimate is valid

$$\mu \cdot e^{-2\sqrt{2\mu}\cos u_0(\mu)} < \lambda_0^+(\mu) < 2\mu(1 + \sqrt{2})e^{-2\sqrt{2\mu}\sin u_0(\mu)}$$

where

$$u_0(\mu) > u_0^*(\mu) > 0, \quad \forall \mu > \mu_0, \quad \lim_{\mu \to +\infty} u_0(\mu) = \frac{\pi}{4}$$

$$\left.\frac{\partial F}{\partial u}\right|_{n=u_0^*} = 0, \qquad 1 - \sqrt{2\mu_0} th\sqrt{2\mu_0} = 0$$

b) we will name the quantity μ_0 the threshold, it is the root of the equation
c)

$$1 - \sqrt{2\mu} th\sqrt{2\mu} = 0$$

for it the assessment holds: 0,5 <μ_0 <0,72, see. ([18].,p.33)
From the system of equations (*) and formula (10), see. P.10 assuming

$$a_n = K_n ch\sqrt{\mu + \lambda_n}, b_n = K_n sh\sqrt{\mu - \lambda_n},$$

we will find the eigenfunctions of the boundary value problem (8) - (9)

$$v_n(\mu, \lambda_n, x) = K_n\left[ch\sqrt{\mu + \lambda_n} sh\sqrt{\mu - \lambda_n} x + sh\sqrt{\mu - \lambda_n} ch\sqrt{\mu - \lambda_n} x\right].$$

Lemma 6. If $\mu \geq 1$, then the spectral task

$$v''(x) - \mu v(x) = \lambda v(-x), \quad x \in (-1,1] \tag{8}$$
$$v(-1) = 0, \quad v'(-1) = 0 \tag{9}$$

has complete and orthogonal system of eigenvectors:

$$v_n(\mu, \lambda_n, x) = K_n\left[ch\sqrt{\mu + \lambda_n} sh\sqrt{\mu - \lambda_n} x + sh\sqrt{\mu - \lambda_n} ch\sqrt{\mu - \lambda_n} x\right], \qquad n = 0,1,2, ...$$

corresponding to the real eigenvalues $\lambda_n(\mu)$, $n = 0,1,2, ...$, which are distributed as follows:
a) negative:

$$-[\mu + (n\pi + \frac{\pi}{2})^2] < \lambda_n^-(\mu) < -\left[\mu + \left(n\pi + \frac{\pi}{4}\right)^2\right], \quad n = 0,1,2 ...$$

b) zero:

$$\mu \cdot e^{-2\sqrt{2\mu}\cos u_0(\mu)} < \lambda_0^+(\mu) < 2\mu(1 + \sqrt{2})e^{-2\sqrt{2\mu}\sin u_0(\mu)}$$

where

$u_0(\mu) > u_0^*(\mu) > 0, \quad \forall \mu > \mu_0, \quad \lim_{\mu\to+\infty} u_0(\mu) = \frac{\pi}{4}$ (monotonically)

$$\left.\frac{\partial F}{\partial u}\right|_{u=u_0^*(\mu)} = 0, \ \ 1-\sqrt{2\mu_0}th\sqrt{2\mu_0} = 0, \quad 0,5 < \mu_0 < 0,72$$

c) positive:

$$\mu + (n\pi)^2 < \lambda_n^+(\mu) < \mu + (n\pi + \frac{\pi}{2})^2, \ \ n = 1,2 \dots$$

g) the interval $[-\mu, 0]$ remains free of eigenvalues where $\mu \geq 0$;

4.3. The proofs of theorems.

Let's start with Theorem 3. Assuming

$$u_{mn}(x,y) = \sin my \cdot v_{mn}(x), \ \ m = 1,2,\dots.$$

We divide the variables of the equation

$$u_{xx} + u_{yy} = \lambda u(-x,y), \quad x \in (-1,1], \ \ y\epsilon(0,\pi)$$

result for $u_{mn}(x,y)$ we obtain the spectral problem

$$v''_{mn} - m^2 v_{mn}(x) = \lambda_{mn} v_{mn}(-x), \ \ m = 1,2,\dots$$
$$v_{mn}(-1) = 0, \ \ v'_{mn}(-1) = 0.$$

By virtue of the proven Lemma 6, eigenfunctions of the spectral problem :$\{v_{mn}(x)\}$, $m = 1,2,\dots, n = 0,1,2,\dots$ form a complete orthogonal system in the space $L^2(-1,1)$. Therefore, after normalization, they form an orthonormal basis of the space.

Lemma 7. If the system $\{\varphi_m(y)\}$, $m = 1,2,\dots$ is an orthonormal basis of the space $L^2(0,\pi)$, and the system $\{\psi_{mn}(x)\}, m = 1,2,\dots; \ n = 0,1,2,\dots$ for each fixed value of m is an orthonormal basis of the space $L^2(-1,1)$, then the system

$$u_{mn}(x,y) = \psi_{mn}(x)\cdot\varphi_m(y), \quad m = 1,2,\dots; n = 0,1,2,\dots$$

is an orthonormal basis of the space $L^2(\Omega)$, where $\Omega = [-1,1] \times [0,\pi]$. See the proof [14].

In our case,

$$\varphi_m(y) = \sqrt{\frac{2}{\pi}}\sin my, \ \ \psi_{mn}(x) = K_{mn}\cdot v_{mn}(x), \ \ m = 1,2,\dots.; n = 0,1,2,\dots,$$

whereK_{mn} are the normalization coefficients, hence eigenfunctions

$$u_{mn}(x,y) = K_{mn}\sin my v_{mn}(x), \qquad m = 1,2,\dots; n = 0,1,2,\dots$$

of the boundary value problem

$$u_{xx} + u_{yy} = \lambda u(-x,y), \quad x \in (-1,1], \qquad y \in (0,\pi)$$
$$u|_{y=0} = 0, \ \ u|_{y=\pi} = 0,$$
$$u|_{x=-1}, \ \left.\frac{\partial u}{\partial x}\right|_{x=-1} = 0,$$

After normalization form an orthonormal basis of the space $L^2(\Omega)$. Theorem 3 is proved. Theorem 2 is a consequence of Lemma 6, when $\mu = m^2$, $m = 1,2,\dots$ Theorem 1 follows from Lemma 1 and Theorem 3.

5. Conclusions.

The operator $\bar{\mathrm{A}}$ has a spectral hatch

$(-\mu_0, 0)$, where0,5 $< \mu_0 <$ 0,72;

1) If $f(x,y) \in L^2(\Omega)$, then the solution of (1) - (3) exists if and only if
2)

$$\sum_{m=1}^{\infty}\sum_{n=0}^{+\infty}\frac{|(Sf,u_{mn})|^2}{\lambda_{mn}^2} < +\infty,$$

where$Sf(x,y) = f(-x,y)$.

2) There is regularizing family of tasks, which has the form
$\bar{A}u + \mu u = Sf$, где $\quad 0 < \mu < \mu_0$.

Proof of paragraph 3).
Let $0 < \mu < \mu_0$, we estimate the resolution of $(\bar{A} + \mu I)^{-1}$. It's obvious that
$\bar{A}u_{mn} + \mu u_{mn} = (\lambda_{mn} + \mu)u_{mn}$,therefore$(\bar{A} + \mu I)^{-1}u_{mn} = \frac{u_{mn}}{\lambda_{mn}+\mu}$.

$$\begin{aligned} u_\mu(x,y) &= [\bar{A} + \mu I]^{-1}Sf \\ &= \sum\nolimits_{m,n}([\bar{A} + \mu I]^{-1}Sf, u_{mn})u_{mn} \\ &= \sum\nolimits_{m,n}(Sf, [\bar{A} + \mu I]^{-1}, u_{mn})u_{mn} = \sum_{m,n}\frac{(Sf,u_{mn})}{\mu + \lambda_{mn}}u_{mn}. \end{aligned}$$

Next, we estimate the distance from the point $-\mu$ to the spectrum of the operator$\bar{A}$.
a) For the negative eigenvalues of the operator$\bar{A}$ thefollowing inequality holds

$$-m^2 - (n\pi)^2 < \lambda_{mn}^- < -m^2 - (n\pi - \frac{\pi}{2})^2, m,n = 1,2,...,$$

Therefore
$-\infty + \infty$

$-\infty + \infty$

$\lambda_{mn}^- - 1 - (\frac{\pi}{2})^2 - 1 - \mu_0 - \mu \quad 0$

since, $0 < \mu < \mu_0 < 0{,}72$,then $0{,}72 < -\mu_0 < -\mu < 0$;

$$|\lambda_{mn}^- + \mu| > \left|-1 - (\frac{\pi}{2})^2 + \mu\right| > \left|-1 - (\frac{\pi}{2})^2 + \mu_0\right| = \left|1 + (\frac{\pi}{2})^2 - \mu_0\right| = 1 - \mu_0 + (\frac{\pi}{2})^2;$$

b) to "zero" eigenvalues $\lambda_{mn}^+ > 0$, we have

$$|-\mu - \lambda_{mn}^+| > |-\mu - 0| = |-\mu| = \mu > 0;$$

For positive eigenvalues the following inequalities hold

$$m^2 + (n\pi)^2 < \lambda_{mn}^+ < m^2 + (n\pi + \frac{\pi}{2})^2, \quad m,n = 1,2,...$$

$-\infty + \infty$

$-1 - \mu_0 - \mu \quad 0 \quad 1 \quad 1 + \pi^2 \quad \lambda_{mn}^+$

$$|-\mu - \mu_{mn}^+| \geq \mu + \lambda_{mn}^+ > \mu + 1 + \pi^2 > 1 + \pi^2.$$

Based on these inequalities, we estimate the resolventof$\bar{A}$, at $\lambda = -\mu_0$.
From the equation,

$$(\bar{A} + \mu I)u_\mu = Sf,$$

we have

$$u_\mu(x,y) = [\bar{A} + \mu I]^{-1} Sf = \sum_{m,n} \frac{(Sf, u_{mn})}{\lambda_{mn} + \mu} u_{mn} = \sum_{m,n} \frac{(Sf, u_{mn})}{\lambda_{mn}^- + \mu} u_{mn} + \sum_{m=1}^{\infty} \frac{(Sf, u_{mn}) u_{mn}}{\lambda_{mn}^- + \mu} += \sum_{m,n} \frac{(Sf, u_{mn})}{\lambda_{mn} + \mu} u_{mn};$$

$$\|u_\mu\|^2 = \sum_{m,n} \frac{(Sf, u_{mn})^2}{(\lambda_{mn}^- + \mu)^2} + \sum_{m=1}^{\infty} \frac{(Sf, u_{m0})^2}{(\lambda_{mn}^+ + \mu)^2} + \sum_{m,n} \frac{(Sf, u_{mn})^2}{(\lambda_{mn}^+ + \mu)^2} \le \frac{1}{\left[1 - \mu_0 + (\frac{\pi}{2})^2\right]^2} \sum_{m,n} |(Sf, u_{mn}|^2 + \frac{1}{\varepsilon^2} \sum_{m=1}^{\infty} |(Sf, u_{m0})|^2 + \frac{1}{1+\pi^2} \sum_{m,n} |(sf, u_{mn})|^2 \le K^2(\varepsilon)\|sf\|^2 \le K^2(\varepsilon) \cdot \|f\|^2,$$

where

$$K(\varepsilon) = \max\left\{\frac{1}{1-\mu_0+(\frac{\pi}{2})^2}, \frac{1}{\varepsilon^2}, \frac{1}{1+\pi^2}\right\}.$$

Assume that for a given $f \in L^2(\Omega)$ there exists a solution of equation
$\bar{A}u = Sf$.
then

$$\lim_{\mu \to 0} \|u_\mu - u\| = 0.$$

Indeed,

$$\|u_\mu(x,y) - u(x,y)\|^2 = \sum_{m,n} \left(\frac{1}{\lambda_{mn} + \mu} - \frac{1}{\lambda_{mn}}\right)^2 |(Sf, u_{mn})|^2 \le \sum_{m,n} \frac{\mu^2}{(\lambda_{mn} + \mu)^2} |(Sf, u_{mn})|^2$$
$$\le \sum_{m,n} \frac{\mu^2}{(\lambda_{mn}^- + \mu)^2 (\lambda_{mn}^-)^2} |(Sf, u_{mn})|^2 + \sum_{m,n} \frac{\mu^2}{(\lambda_{m0}^+ + \mu)^2 (\lambda_{m0}^+)^2} |(Sf, u_{m0})|^2 + \sum_{m,n}^{+\infty} \frac{\mu^2}{(\lambda_{mn}^+ + \mu)^2} |(Sf, u_{mn})|^2$$
$$\le \frac{\mu^2}{\left[1 - \mu_0 + (\frac{\pi}{2})^2\right]^2} \sum_{m,n} \frac{|(Sf, u_{mn}|^2}{(\lambda_{mn}^-)^2} + \frac{\mu^2}{(1+\pi^2)^2} \sum_{m,n} \frac{|(Sf, u_{mn}|^2}{(\lambda_{mn}^+)^2} + \sum_{m=1}^{\infty} \frac{\mu^2}{(\lambda_{m0}^+ + \varepsilon)^2 (\lambda_{m0}^+)^2} |(Sf, u_{m0})|^2 <$$

$$< \mu^2 \left[\frac{1}{\left[1 - \mu_0 + \left(\frac{\pi}{2}\right)^2\right]^2} + \frac{1}{(1+\pi^2)^2} \right] \|f\|^2$$

$$+ \sum_{m=1}^{\infty} \frac{\mu^2 |(Sf, u_{m0}|^2}{\left(\lambda_{m0}^+\right)^2 (\lambda_{m0}^+ + \mu)} < \frac{2\mu^2}{\left[1 - \mu_0 + \left(\frac{\pi}{2}\right)^2\right]^2} + \sum_{m=1}^{\infty} \frac{\mu^2}{\left(\left(\lambda_{m0}^+\right)^2 \lambda_{m0}^+ + \mu\right)^2} |(Sf, u_{m0})|^2$$

$$\leq \left| \begin{matrix} \lambda_{10}^+ > \lambda_{20}^+ > \cdots > \lambda_{N0}^+ \\ \frac{\mu}{\lambda_{m0}^+ + \mu} < 1 \end{matrix} \right|$$

$$\leq \frac{2\mu^2}{\left[1 - \mu_0 + \left(\frac{\pi}{2}\right)^2\right]^2} \|f\|^2$$

$$+ \frac{\mu^2}{\left(\lambda_N^+\right)^+} \sum_{m=1}^{N} |(Sf, u_{m0})|^2$$

$$+ \sum_{N+1}^{\infty} \frac{|(Sf, u_{m0})|^2}{\left(\lambda_{m0}^+\right)^2}$$

$$< \frac{2\mu^2 \|f\|^2}{\left[1 - \mu_0 + \left(\frac{\pi}{2}\right)^2\right]^2} + \frac{\mu^2}{\left(\lambda_N^+\right)^+} \|f\|^2$$

$$+ \sum_{N+1}^{\infty} \frac{|(Sf, u_{m0})|^2}{\left(\lambda_{m0}^+\right)^2} \leq \frac{3\mu^2}{\left(\lambda_N^+\right)^2} \|f\|^2 + \sum_{N+1}^{\infty} \frac{|(Sf, u_{m0})|^2}{\left(\lambda_{m0}^+\right)^2}.$$

By our assumption$\sum_{m=1}^{\infty} \frac{|(Sf,u_{m0})|^2}{\left(\lambda_{m0}^+\right)^2} < +\infty$, therefore for any $\varepsilon > 0$there exists a number $N(\varepsilon)$such that for all $M > N(\varepsilon)$ the inequality holds

$\sum_{M}^{+\infty} \frac{|(Sf,u_{m0})|^2}{\left(\lambda_{m0}^+\right)^2} < \frac{\varepsilon^2}{2}$.

Then, at a fixed $N(\varepsilon)$, there exists a number $\delta > 0$such that for all $0 \leq \mu < \delta$the following inequality holds

$$\frac{3\mu^2 \|f\|^2}{(\lambda_N^+)^2} < \frac{\varepsilon^2}{2}$$

Consequently, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$\left\|u_\mu - u\right\|^2 < \varepsilon^2$ i.e. $\left\|u_\mu - u\right\| < \varepsilon$for all $0 \leq \mu < \delta$, as required.

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ЛАПЛАС ТЕҢДЕУІНІҢ КОШИЛІК ЕСЕБІНІҢ СПЕКТРӘЛДІК ТАРАЛЫМЫ

Аннотация. Лаплас теңдеуіне қойылған Кошидің есебінің шешімі бар екені анықталып,оның Крейннің кеңістігіндегі спектрәлді таралымы алынды. Сонан соң резөлвентасы арқылы есептің жөнге келетіні көрсетілді.

Түйін сөздер: Кошидің есебі, Лапластың теңдеуі, ауытқыған аргумент, жалылық, әсіре үзіксіздік, Гилберт пен Шмидтің теоремасы.

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СПЕКТРАЛЬНОЕ РАЗЛОЖЕНИЕ РЕШЕНИЯ ЗАДАЧИ КОШИ ДЛЯ УРАВНЕНИЯ ЛАПЛАСА

Аннотация.В пространстве Крейна, получено спектральное разложение решении задачи Коши уравнения Лапласа,и произведена регуляризация задачи, с помощью резольвенты соответствующего оператора.

Ключевые слова: Задача Коши, уравнение Лапласа, некорректность, спектр, отклояющиеся аргумент, самосопряженность, компактность, теорема Гильберта-Шмидта.

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