N E W S
OF THE NATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
PHYSICO-MATHEMATICAL SERIES
ISSN 1991-346X **https://doi.org/10.32014/2019.2518-1726.56**
Volume 5, Number 327 (2019), 19 – 38

UDK 517.929

A.Sh. Shaldanbayev[1], S.M. Shalenova[2], M.B. Ivanova[3], A.A. Shaldanbayeva[4]

[1]Silkway International University, Shymkent, Kazakhstan;
[2]Yessenov University; Aktau, Kazakhstan;
[3]South Kazakhstan medical Academy, Shymkent, Kazakhstan;
[4]Regional social-innovative University, Shymkent, Kazakhstan
shaldanbaev51@mail.ru, salima_shalenova@mail.ru, marina-iv@mail.ru, altima_a@mail.ru

ON SPECTRAL PROPERTIES OF A BOUNDARY VALUE PROBLEM OF THE FIRST ORDER EQUATION WITH DEVIATING ARGUMENT

Abstract. In this paper, we study spectral properties of a boundary value problem of a first order differential equation with constant coefficients and deviating argument; the deviation is present at the highest term of the equation, and it cannot be transferred to the lower terms of the equation without an additional condition. By spectral properties, we mean completeness and basic properties of a system of eigenfunctions and associated functions of a boundary value problem, as well as Volterra properties.

Keywords: equation with deviating argument, completeness, basic property, Volterra property, Gaal's formula, Lidsky's theorem, Sturm – Liouville operator, Riesz basis.

1. Introduction. The first works on the theory of differential equations with involution are found in the scientific literature of the nineteenth century. This is the extensive work of Babbage [1] from 1816. The work consists of several parts. Algebraic and transcendental equations with involution, integral and differential equations containing an involution are considered. At present, mathematicians from many countries are studying differential equations with involution. The reference devoted to the study of differential equations with involution is quite extensive. An extensive bibliography contains monographs by D. Przevorska-Rolevicz [2], J. Wiener [3]. Jack K. Hale Sjoerd M. Verduyn Lunel [4].

Partial differential equations with involution arise in mathematical models of population dynamics, ecology and physiology. A series of papers by S. Busenberg [6], J.M. Cushing [7-8] and others, devoted to mathematical modeling in population theory (biology), suggest the need for deep research on the analytical theory of differential equations with involution. In the work of J. Wiener [3, p. 264], attempts were made to apply the method of variable separation to partial differential equations with involution. In this case, solution is sought as a series in eigenfunctions. Conditions on existence of unbounded solutions of the considered problems are obtained, as well as the condition for a series to diverge in terms of eigenfunctions. Among the studies of recent years we can note the work of W.Watkins [9-10], which deals with solvability of one-dimensional differential equations with involution, and A.P. Khromov and his followers [11–12], which consider questions of solvability of integral and partial differential equations with involution.

Method of variable separation for solving partial differential equations is based on the spectral theory of one-dimensional differential operators. Spectral theory of self-adjoint and non-self-adjoint ordinary differential operators, which originated in depths of the equations of mathematical physics and began with classical works of Sturm, Liouville, Steklov and others, has received a fairly complete development over the past century. Spectral theory of self-adjoint ordinary differential operators is almost complete. In the field of spectral theory of non-self-adjoint ordinary differential operators, substantial results on completeness and basicity of eigenfunctions and associated functions are obtained by M.V. Keldysh [13], V.A. Il'in [14-19], M. Otelbaev [20], A.A. Shkalikov [21], Radzievsky [22] and many other mathematicians.

Theory of basicity of systems of eigenfunctions and associated functions of non-self-adjoint ordinary differential operators, proposed by V.A. Ilyin, received rapid development. Review papers [23-24] give a fairly complete picture of development of the basicity theory by V.A. Il'in.

Compared with the spectral theory of ordinary differential operators, the spectral theory of one-dimensional differential operators with involution is in its infancy. Apparently, the first works on the spectral theory of one-dimensional differential operators with involution were carried out on initiative of T.Sh. Kalmenov [25-29] in the past decade of this century. These studies were continued in the cycle of works by M.A. Sadybekov and A.M. Sarsenby [30-35]. Over the past decade, interest of researchers to differential equations with involutions has noticeably increased, as evidenced by the publications [36–52]. The bases theory is described in [54–55] in detail.

In this paper we continue the studies begun in [28], a brief summary of this paper was announced in [53].

Formulation of the problem. We investigate the boundary value problem

$$Ly = ay'(x) + by'(1-x) = \lambda y(x), x \in (0,1), \tag{1.1}$$

$$\alpha y(0) + \beta y(1) = 0, \tag{1.2}$$

where a, b, α, β are arbitrary, previously known constants, and λ is a spectral parameter, $y(x)$ is a desired function from the class $C^1(0,1) \cap C[0,1]$.

Let operator P be defined by the following formula

$$Py(x) = ay(x) + by(1-x),$$

then

$$P^{-1}v(x) = \frac{(aI - bS)v(x)}{a^2 - b^2},$$

where $Sv(x) = v(1-x)$, I is unit operator.

Indeed,

$$PP^{-1}v(x) = (aI + bS)\frac{(aI - bS)v(x)}{a^2 - b^2} =$$
$$= \frac{a^2 I - abS + baS - b^2 S^2}{a^2 - b^2} = \frac{(a^2 - b^2)I}{a^2 - b^2} = I.$$

Consequently, if $a^2 - b^2 \neq 0$, then the problem (1.1) - (1.2) takes the following form:

$$Ly = P\frac{d}{dx}y(x) = f(x), x \in (0,1),$$
$$\alpha y(0) + \beta y(1) = 0.$$

$$\begin{cases} y'(x) = \lambda P^{-1}y(x), \\ \alpha y(0) + \beta y(1) = 0; \end{cases} \begin{cases} y'(x) = \lambda \dfrac{aI - bS}{a^2 - b^2} y(x), \\ \alpha y(0) + \beta y(1) = 0. \end{cases}$$

It is a generalized spectral problem.

Let $Qy(x) = ay(x) + by(1-x)$, then

$$Q^{-1}v(x) = \frac{(aI - bS)v(x)}{\alpha^2 - \beta^2}.$$

Supposing $Qy(x) = z(x)$, we have

$$y(x) = Q^{-1}z(x) = \frac{aI - bS}{\alpha^2 - \beta^2}z(x) = \frac{\alpha z(x) - \beta z(1-x)}{\alpha^2 - \beta^2},$$

$$y'(x) = \frac{\alpha z'(x) + \beta z'(1-x)}{\alpha^2 - \beta^2} = \frac{Qz'(x)}{\alpha^2 - \beta^2};$$

Then our problem takes the following form:

$$\frac{Qz'(x)}{\alpha^2 - \beta^2} = \lambda \frac{aI - bS}{a^2 - b^2} Q^{-1} z(x), => \ Qz'(x) = \lambda \frac{\alpha^2 - \beta^2}{a^2 - b^2} Q^{-1} z(x),$$
$$Qz'(x) = \lambda(\alpha^2 - \beta^2) P^{-1} Q^{-1} z(x),$$

$$\begin{cases} z'(x) = \lambda(\alpha^2 - \beta^2) Q^{-1} P^{-1} Q^{-1} z(x), \\ z(0) = 0. \end{cases} \quad (1.3), (1.4)$$

It is a generalized spectral Cauchy problem.

Remark 1.1. If $(\alpha^2 - \beta^2)(a^2 - b^2) \neq 0$, then the problem (1.1)-(1.2) is equivalent to the generalized spectral Cauchy problem (1.3)-(1.4).

2. Research Methods.

2.1. Solvability. If $y_n(x) = \sin n\pi x$, then

$$y_n(1-x) = (-1)^{n+1} y_n(x), y_{2m}(1-x) = -y_{2m}(x),$$
$$y_{2m-1}(x) = y_{2m-1}(1-x);$$

moreover $y_n(0) = 0, y_n(1) = 0$.

а) $y_{2m}(1-x) + y_{2m}(x) = 0, => y'_{2m}(x) - y'_{2m}(1-x) = 0;$

б) $y_{2m-1}(x) - y_{2m-1}(x) = 0, => y'_{2m-1}(x) + y'_{2m-1}(x) = 0.$

Lemma 2.1. If $Ly_0 = 0, y_0(x) \not\equiv 0$, then

$$(a^2 - b^2)(\alpha + \beta) = 0. \ (2.1)$$

Proof. From the equation (1.1) we have

$$\begin{cases} ay'(x) + by'(1-x) = 0, \\ by'(x) + ay'(1-x) = 0; \end{cases} =>$$

а) or $a^2 - b^2 = 0$, and $y'(x) \not\equiv 0$;

б) or $a^2 - b^2 \neq 0$, and $y'(x) \equiv 0$.

If $a = b \neq 0$, then the problem has the form

$$\begin{cases} y'(x) + y'(1-x) = 0, \\ \alpha y(0) + \beta y(1) = 0. \end{cases}$$

Functions $y_{2m-1}(x) = \sin(2m-1)\pi x, m = 1,2, \ldots$ are solutions of this problem, therefore, in this case the point $\lambda_0 = 0$ is an infinitely multiple eigenvalue of the boundary value problem (1.1)-(1.2).

If $b = -a \neq 0$, then the boundary value problem (1.1)-(1.2) takes the following form

$$\begin{cases} y'(x) - y'(1-x) = 0, \\ \alpha y(0) + \beta y(1) = 0. \end{cases}$$

Solution of this problem is the following functions

$$y_{2m}(x) = \sin 2m\pi x, m = 1,2, \ldots$$

therefore, also in this case the point $\lambda_0 = 0$ is infinitely multiple eigenvalue of the boundary value problem (1.1) - (1.2).

If $a^2 - b^2 \neq 0$, then $y'(x) \equiv 0, y(x) = C - const; =>$
$$\alpha \cdot C + \beta \cdot C = 0, (\alpha + \beta) \cdot C = 0, => \ \alpha + \beta = 0,$$

since $C \neq 0$.

Let $(a^2 - b^2)(\alpha + \beta) = 0$, then

а) or $\alpha + \beta = 0$, or $\alpha + \beta \neq 0$, then $a^2 - b^2 = 0$.

If $\alpha+\beta=0$, then the function $y_0(x)=C-const$ is an eigenfunction and $\lambda_0=0$ is an eigenvalue.

If $a^2-b^2=0$, then as we have already noted, in the case $a=b$ the functions $y_{2m-1}(x)=\sin(2m-1)\pi x, m=1,2,\ldots$ are eigenfunctions, and $\lambda_0=0$ is eigenvalue of the infinitely multiple. In the case $b=-a$ the functions $y_{2m}(x)=\sin 2m\pi x$ are eigenfunctions, and $\lambda_0=0$ is infinitely multiple eigenvalue.

Lemma 2.2. Operator L is invertible if and only if

$$(a^2-b^2)(\alpha+\beta)\neq 0, \tag{2.2}$$

where

$$Ly=ay'(x)+by'(1-x), \tag{1.1}$$

$$\alpha y(0)+\beta y(1)=0. \tag{1.2}$$

Proof. If $(a^2-b^2)(\alpha+\beta)=0$, then in the case $a^2-b^2=0$, $\lambda_0=0$ is eigenvalue, thus the operator L is not invertible, and in the case $a^2-b^2\neq 0$, $\lambda_0=0$ is a simple eigenvalue, therefore the operator L is again non-invertible.

Inversely, if operator L is non-invertible, then there exists a function $y_0(x)\not\equiv 0$ such that $Ly_0=0$, then due to Lemma 2.1, we have the equality (2.1).

2.2. About inverse operator.

$$Ly=ay'(x)+by'(1-x)=f(x), x\in(0,1), \tag{1.1}$$

$$\alpha y(0)+\beta y(1)=0, |\alpha|+|\beta|\neq 0. \tag{1.2}$$

Let $Tu(x)=au(x)+bu(1-x)=(aI+bS)u(x)$,

where $Su(x)=u(1-x)$, then

$$T^{-1}v(x)=\frac{aI-bS}{\alpha^2-\beta^2}v(x).$$

Indeed,

$$TT^{-1}v(x)=(aI+bS)\frac{(aI-bS)v(x)}{a^2-b^2}=\frac{(a^2I-abS+baS-b^2S^2)}{a^2-b^2}v(x)=v(x).$$

The problem (1.1)-(1.2) takes the following form

$$\begin{cases} T\dfrac{d}{dx}y(x)=f(x), x\in(0,1),\\ \alpha y(0)+\beta y(1)=0.\end{cases}$$

Consequently,

$$y'(x)=T^{-1}f(x), \Rightarrow$$

$$y(x)=y(0)+\int_0^x T^{-1}f(t)dt, y(1)-y(x)=\int_x^1 T^{-1}f(t)dt,$$

$$y(x)=y(1)-\int_x^1 T^{-1}f(t)dt, \Rightarrow$$

$$(\alpha+\beta)y(x)=\alpha\int_0^x T^{-1}f(t)dt-\beta\int_x^1 T^{-1}f(t)dt,$$

$$y(x) = L^{-1}f(x) = \frac{\alpha}{\alpha+\beta}\int_0^x T^{-1}f(t)dt - \frac{\beta}{\alpha+\beta}\int_x^1 T^{-1}f(t)dt =$$

$$= \frac{\alpha}{\alpha+\beta}\int_0^1 \theta(x-t)T^{-1}f(t)dt - \frac{\beta}{\alpha+\beta}\int_0^1 и(x-t)T^{-1}f(t)dt =$$

$$= \int_0^1 \frac{\alpha\theta(x-t) - \beta\theta(t-x)}{\alpha+\beta} T^{-1}f(t)dt =$$

$$= \int_0^1 \frac{\alpha\theta(t-x) - \beta\theta(t-x)}{\alpha+\beta} \cdot \frac{af(t) - bf(1-t)}{a^2-b^2}dt =$$

$$= \frac{a}{(a^2-b^2)(\alpha+\beta)}\int_0^1 [\alpha\theta(x-t) - \beta\theta(t-x)]f(t)dt -$$

$$- \frac{b}{(a^2-b^2)(\alpha+\beta)}\int_0^1 [\alpha\theta(x-t) - \beta\theta(t-x)]f(t)dt =$$

$$\frac{a}{(a^2-b^2)(\alpha+\beta)}\int_0^1 [\alpha\theta(x-t) - \beta\theta(t-x)]f(t)dt -$$

$$- \frac{b}{(a^2-b^2)(\alpha+\beta)}\int_0^1 [\alpha\theta(x-1+t) - \beta\theta(1-t-x)]f(t)dt =$$

$$= \int_0^1 \frac{a[\alpha\theta(x-t) - \beta\theta(t-x)] - b[\alpha\theta(x-1+t) - \beta\theta(1-t-x)]}{(a^2-b^2)(\alpha+\beta)} \cdot$$

$$\cdot f(t)dt = \int_0^1 K(x,t)f(t)dt,$$

where

$$K(x,t) = \frac{a[\alpha\theta(x-t) - \beta\theta(t-x)] - b[\alpha\theta(x-1+t) - \beta\theta(1-t-x)]}{(a^2-b^2)(\alpha+\beta)}.$$

We have proved the following theorem.

Lemma 2.3. If

$$(a^2-b^2)(\alpha+\beta) \neq 0$$

then the inverse operator L^{-1} exists and has the form

$$y(x) = L^{-1}f(x) = \int_0^1 K(x,t)f(t)dt,$$

where

$$K(x,t) = \frac{a[\alpha\theta(x-t) - \beta\theta(t-x)] - b[\alpha\theta(x-1+t) - \beta\theta(1-t-x)]}{(a^2-b^2)(\alpha+\beta)}.$$

2.3. Criteria about Volterra property.

We consider the following boundary value problem

$$Ly = ay'(x) + by'(1-x) = f(x), x \in (0,1), \quad (1.1)$$

$$\alpha y(0) + \beta y(1) = 0, \quad (1.2)$$

in the space $L^2(0,1)$, where a, b, α, β are arbitrary complex constants, satisfying the condition

$$(|a| + |b|)(|\alpha| + |\beta|) \neq 0, b \neq 0 \quad (2.1)$$

$f(x)$ is a continuous function on the segment [0,1], $y(x)$ is a unknown, continuously differentiable function.

Definition 2.1. If the boundary value problem

$$Ly = ay'(x) + by'(1-x) - \lambda y(x) = 0, \quad (1.1)$$

$$\alpha y(0) + \beta y(1) = 0, \quad (1.2)$$

has only trivial solution at any complex λ, then it is called Volterra.

Theorem 2.1. Boundary value problem (1.1) - (1.2) is Volterra if and only if

1) $(a^2 - b^2)(\alpha^2 - \beta^2) \neq 0$; (2.2)

2) $b(\alpha^2 + \beta^2) + 2\alpha\beta a = 0$. (2.3)

Proof.

a) Necessity. Let boundary value problem (1.1) - (1.2) be Volterra, then $(a^2 - b^2)(\alpha + \beta) \neq 0$, otherwise $\lambda_0 = 0$ is eigenvalue, which contradicts the Volterra property of the boundary value problem (1.1) - (1.2). If the condition $(a^2 - b^2)(\alpha + \beta) \neq 0$ holds, then there exists inverse operator L^{-1}, which has the form

$$L^{-1}f(x) = \int_0^1 K(x,t)f(t)dt,$$

where

$$K(x,t) = \frac{a[\alpha\theta(x-t) - \beta\theta(t-x)] - b[\alpha\theta(x-1+t) - \beta\theta(1-t-x)]}{(a^2-b^2)(\alpha+\beta)}.$$

If the operator L does not any nonzero eigenvalues, then the operator L^{-1} also does not have nonzero eigenvalues, consequently, kernel operator $(L^{-1})^2$ does not have nonzero eigenvalues. Then, by the Lidsky theorem [59], we get

$$SpL^{-2} = 0.$$

Now we calculate the left part of this formula

$$L^{-2}f(x) = \int_0^1 K^2(x,t)f(t)dt,$$

where

$$K^2(x,t) = \int_0^1 K(x,\xi)K(\xi,t)d\xi.$$

It is known that (by the Gaal theorem) [60]

$$SpL^{-2} = \int_0^1 K^2(x,x)\,dx,$$

thus

$$SpL^{-2} = \int_0^1\int_0^1 K(x,\xi)K(\xi,t)d\xi dx.$$

It remains to calculate this double integral:

$$K(x,\xi)K(\xi,t) = ?$$

$$K(x,\xi)=\frac{a[\alpha\theta(x-\xi)-\beta\theta(\xi-x)]-b[\alpha\theta(x-1+\xi)-\beta\theta(1-\xi-x)]}{(a^2-b^2)(\alpha+\beta)}=\frac{K[1]-K[2]}{(a^2-b^2)(\alpha+\beta)};$$

$$K(\xi,x)=\frac{a[\alpha\theta(\xi-x)-\beta\theta(x-\xi)]-b[\alpha\theta(\xi-1+x)-\beta\theta(1-x-\xi)]}{(a^2-b^2)(\alpha+\beta)}=\frac{K[3]-K[2]}{(a^2-b^2)(\alpha+\beta)}.$$

$$K[1]\cdot K[3]=-\alpha\beta\theta(x-\xi)-\beta\alpha\theta(\xi-x)=-\alpha\beta[\theta(x-\xi)+\theta(\xi-x)]=-\alpha\beta;$$

$$K[1]\cdot K[2]=\alpha^2\theta(x-\xi)\cdot\theta(x+\xi-1)-\alpha\beta\theta(x-\xi)\theta(1-x-\xi)-$$
$$-\beta\alpha\theta(\xi-x)\theta(x+\xi-1)+\beta^2\theta(\xi-x)\cdot\theta(1-x-\xi);$$

$$K[2]\cdot K[3]=\alpha^2\theta(x+\xi-1)\cdot\theta(\xi-x)-\alpha\beta\theta(x+\xi-1)\theta(x-\xi)-$$
$$-\beta\alpha\theta(1-x-\xi)\theta(\xi-x)+\beta^2\theta(1-x-\xi)\cdot\theta(x-\xi);$$

$$K[2]\cdot K[2]=\alpha^2\theta(x+\xi-1)+\beta^2\theta(1-x-\xi);$$

Consequently,

$$K(x,\xi)=\frac{aK[1]-bK[2]}{(a^2-b^2)(\alpha+\beta)},K(\xi,x)=\frac{aK[3]-bK[2]}{(a^2-b^2)(\alpha+\beta)},$$

$$K(x,\xi)\cdot K(\xi,x)=\frac{a^2K[1]\cdot K[3]-abK[1]\cdot K[2]-baK[2]\cdot K[3]+b^2K^2[2]}{(a^2-b^2)(\alpha+\beta)}=$$

$$=a^2(-\alpha\beta)-ab[\alpha^2\theta(x-\xi)\cdot\theta(x+\xi-1)-\alpha\beta\theta(x-\xi)\theta(1-x-\xi)-$$

$$-\beta\alpha\theta(\xi-x)\theta(x+\xi-1)+\beta^2\theta(\xi-x)\cdot\theta(1-x-\xi)]-$$

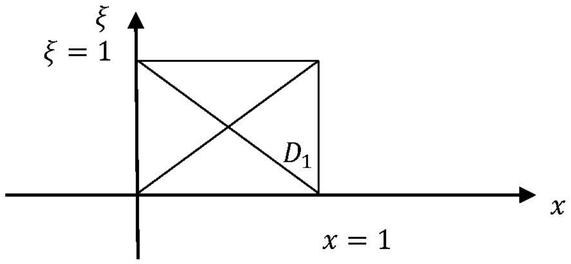
$$-ba[\alpha^2\theta(x+\xi-1)\cdot\theta(\xi-x)-\alpha\beta\theta(x+\xi-1)\theta(x-\xi)-$$

$$-\beta\alpha\theta(1-x-\xi)\theta(\xi-x)+\beta^2\theta(1-x-\xi)\cdot\theta(x-\xi)]+$$

$$+b^2[\alpha^2\theta(x+\xi-1)+\beta^2\theta(1-x-\xi)]\,/\,(a^2-b^2)(\alpha+\beta)^2;$$

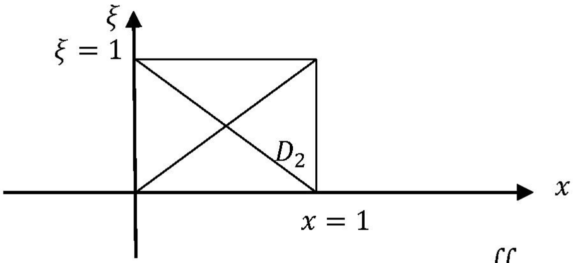
We calculate the following integrals:

1) $\int_0^1\int_0^1 и(x-\xi)\theta(x+\xi-1)d\xi dx=\left|\begin{matrix}x-\xi>0,\\ x+\xi-1>0,\end{matrix}\;=>x>\xi>1-x\right|$



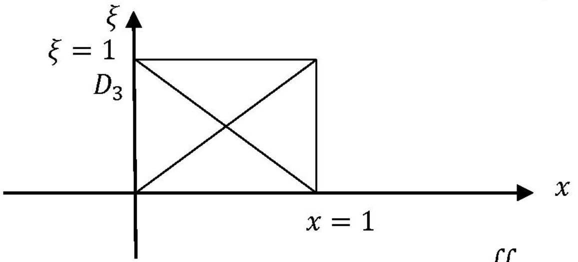
$$=\iint_{D_1}d\xi dx=\frac{1}{4};$$

2) $\int_0^1 \int_0^1 \theta(x-\xi)\theta(1-x-\xi)d\xi dx = \left| \begin{matrix} x-\xi > 0, \\ 1-x-\xi > 0, \end{matrix} \Rightarrow \begin{matrix} x > \xi, \\ 1-\xi > x > \xi, \end{matrix} \right|$



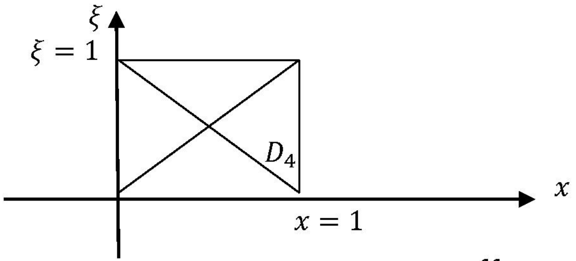
$$= \iint_{D_2} d\xi dx = \frac{1}{4};$$

3) $\int_0^1 \int_0^1 \theta(\xi-x)\theta(x+\xi-1)d\xi dx = \left| \begin{matrix} \xi-x > 0, \\ x+\xi-1 > 0, \end{matrix} \Rightarrow \begin{matrix} \xi > x, \\ \xi > 1-x, \end{matrix} \right|$



$$= \iint_{D_3} d\xi dx = \frac{1}{4};$$

4) $\int_0^1 \int_0^1 \theta(\xi-x)\theta(1-x-\xi)d\xi dx = \left| \begin{matrix} x-\xi > 0, \\ 1-x-\xi > 0, \end{matrix} \right|$



$$= \iint_{D_4} d\xi dx = \frac{1}{4};$$

5) $\int_0^1 \int_0^1 [\alpha^2\theta(x+\xi-1) + \beta^2\theta(1-x-\xi)]d\,\xi dx = ?$

$$\int_0^1 \int_0^1 \alpha^2\theta(x+\xi-1)d\xi dx = \int_0^1 dx \int_{1-x}^{x} \alpha^2 d\xi = \int_0^1 (1-1+x)\alpha^2 dx =$$

$$= \alpha^2 \int_0^1 x dx = \alpha^2 \frac{x^2}{2}\bigg|_0^1 = \frac{\alpha^2}{2};$$

$$\int_0^1 \int_0^1 \beta^2\theta(1-x-\xi)dx d\xi = \beta^2 \int_0^1 dx \int_0^{1-x} d\xi = \beta^2 \int_0^1 (1-x)dx =$$

$$= \beta^2\left(x - \frac{x^2}{2}\right)\bigg|_0^1 = \beta^2\left(1-\frac{1}{2}\right) = \frac{\beta^2}{2};$$

$$SpL^{-2} = \int_0^1 \int_0^1 K(x,\xi)K(\xi,t)d\xi dx =$$

$$= \frac{a^2(-\alpha\beta) - \frac{ab}{4}(\alpha^2 - \alpha\beta - \beta\alpha + \beta^2) - \frac{ba}{4}(\alpha^2 - \alpha\beta - \beta\alpha + \beta^2) + b^2\frac{\alpha^2+\beta^2}{2}}{(a^2-b^2)^2(\alpha+\beta)^2}$$

$$= \frac{-2\alpha\beta a^2 - ab(\alpha-\beta)^2 + b^2(\alpha^2+\beta^2)}{2(a^2-b^2)^2(\alpha+\beta)^2} =$$

$$= \frac{-2\alpha\beta a^2 - ab\alpha^2 + 2\alpha\beta ab - ab\beta^2 + b^2\alpha^2 + b^2\beta^2}{2(a^2-b^2)^2(\alpha+\beta)^2} =$$

$$= \frac{[\alpha^2(b^2-ab) + \beta^2(b^2-ab) + 2\alpha\beta ab - 2\alpha\beta a^2]}{2(a^2-b^2)^2(\alpha+\beta)^2} =$$

$$= \frac{(b^2-ab)(\alpha^2+\beta^2) + 2\alpha\beta(ab-a^2)}{2(a^2-b^2)^2(\alpha+\beta)^2} =$$

$$= \frac{b(b-a)(\alpha^2+\beta^2) + 2\alpha\beta a(b-a)}{2(a^2-b^2)^2(\alpha+\beta)^2} =$$

$$= \frac{(b-a)[b(\alpha^2+\beta^2) + 2\alpha\beta a]}{2(a^2-b^2)^2(\alpha+\beta)^2} = 0.$$

Since $b - a \neq 0$, then $b(\alpha^2+\beta^2) + 2\alpha\beta a = 0$, thus the necessity of the second condition is proved.

If $\beta = -\alpha$, then $b \cdot 2\alpha^2 - 2\alpha^2 \cdot a = 2\alpha^2(b-a) = 0$, since $\alpha \neq 0$, therefore $b - a = 0$, it is impossible, thus $\beta \neq -\alpha$ or $\beta + \alpha \neq 0$.

Similarly, if $\beta = \alpha$, then $b \cdot 2\alpha^2 + 2\alpha^2 \cdot a = 0, 2\alpha^2(b+a) = 0$, since $\alpha \neq 0$, then $b + a = 0$, it is impossible, therefore $\beta - \alpha \neq 0$. Earlier we showed that $(a^2-b^2)(\alpha+\beta) \neq 0$, due to the last condition, we have $(a^2-b^2)(\alpha^2-\beta^2) \neq 0$, t.e. we get the inequality (4.2).

б) Sufficiency.

Let $(a^2-b^2)(\alpha^2-\beta^2) \neq 0$ and $b(\alpha^2+\beta^2) + 2\alpha\beta a = 0$. From the equation (1.1) we have

$$T\frac{d}{dx}y(x) = \lambda y(x), y'(x) = \lambda T^{-1}y(x),$$

$$y'(x) = \lambda\frac{aI - bS}{a^2-b^2}y(x).$$

Differentiating this equation, we obtain

$$y''(x) = \lambda\frac{ay' + by'(1-x)}{a^2-b^2} = \lambda\frac{aI+bS}{a^2-b^2}y'(x) =$$

$$= \lambda^2\frac{aI+bS}{a^2-b^2}\cdot\frac{aI-bS}{a^2-b^2} = \frac{\lambda^2}{a^2-b^2}y(x).$$

Lemma 2.4. If $\lambda \neq 0$ is a eigenvalue of the boundary value problem

$$ay'(x) + by'(1-x) = \lambda y(x), x \in (0,1), \quad (1.1)$$

$$\alpha y(0) + \beta y(1) = 0, \quad (1.2)$$

where $b \neq 0, |\alpha| + |\beta| \neq 0, a^2 - b^2 \neq 0$, then the quantity

$$\mu^2 = \frac{\lambda^2}{a^2-b^2} \quad (2.4)$$

is eigenvalue of the Sturm - Liouville operator

$$y''(x) = \mu^2 y(x), x \in (0,1), \quad (2.5)$$

$$\begin{cases} \alpha y(0) + \beta y(1) = 0, \\ \alpha[ay'(0) + by'(1)] + \beta[ay'(1) + by'(0)] = 0, \end{cases} \quad (2.6)$$

Corollary 2.1. If the Sturm - Liouville problem (2.5)-(2.6) does not have nonzero eigenvalues, then the problem (1.1)-(1.2) also does not them.

Boundary value problem (2.5)-(2.6) does not eigenvalues if and only if

$$\Delta_{24} = 0, \Delta_{14} + \Delta_{32} = 0, \Delta_{13} = 0,$$

where Δ_{ij} are minors of the matrix

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \alpha a + \beta b & 0 & \alpha b + \beta a \end{pmatrix}.$$

It is obvious that

$$\Delta_{24} = 0, \Delta_{14} = \alpha(\alpha b + \beta a), \Delta_{23} = -\beta(\alpha a + \beta / b), \Delta_{13} = 0.$$

Moreover,

$$\Delta_{14} + \Delta_{32} = \alpha(\alpha b + \beta a) + \beta(\alpha a + \beta b) =$$
$$= \alpha^2 b + \alpha\beta a + \beta\alpha a + \beta^2 b = b(\alpha^2 + \beta^2) + 2\alpha\beta a = 0.$$

Consequently, the problem (4.5) - (4.6) is Volterra, i.e. does not have eigenvalues on all complex plane, thus the problem (1.1) - (1.2) also does not have nonzero eigenvalues.

Remark 2.1.

Other proof of sufficiency.

If $\alpha^2 - \beta^2 \neq 0$ and $a^2 - b^2 \neq 0$, then our boundary value problem is equivalent to the generalized Cauchy problem

$$\begin{cases} z'(x) = \lambda(\alpha^2 - \beta^2)Q^{-1}P^{-1}Q^{-1}z(x), \\ z(0) = 0; \end{cases} \quad (2.7), (2.8)$$

where

$$Pu(x) = au(x) + bu(1-x) = (aI + bS)u(x),$$

$$Qu(x) = \alpha u(x) + \beta u(1-x) = (\alpha I + \beta S)u(x),$$

$$Q^{-1} = \frac{\alpha I - \beta S}{\alpha^2 - \beta^2}, P^{-1} = \frac{aI - bS}{a^2 - b^2},$$

$$P^{-1}Q^{-1} = \frac{aI - bS}{a^2 - b^2} \cdot \frac{\alpha I - \beta S}{\alpha^2 - \beta^2} = \frac{a\alpha I - a\beta S - b\alpha S + b\beta S}{(a^2 - b^2)(\alpha^2 - \beta^2)} =$$

$$= \frac{a\alpha I - (a\beta + b\alpha)S + b\beta S}{(a^2 - b^2)(\alpha^2 - \beta^2)} = \frac{(a\alpha + b\beta)I - (a\beta + b\alpha)S}{(a^2 - b^2)(\alpha^2 - \beta^2)};$$

$$Q^{-1}P^{-1}Q^{-1} = \frac{\alpha I - \beta S}{\alpha^2 - \beta^2} \cdot \frac{(a\alpha + b\beta)I - (a\beta + b\alpha)S}{(a^2 - b^2)(\alpha^2 - \beta^2)} =$$

$$= \frac{\alpha(a\alpha + b\beta)I + \beta(a\beta + b\alpha)I - \alpha(a\beta + b\alpha)S - \beta(a\alpha + b\beta)S}{(a^2 - b^2)(\alpha^2 - \beta^2)^2} =$$

$$= \frac{(\alpha^2 a + \alpha\beta b + \beta^2 a + \alpha\beta b)I - (a\alpha\beta + \alpha^2 b + \beta\alpha a + \beta^2 b)S}{(a^2 - b^2)(\alpha^2 - \beta^2)^2} =$$

$$= \frac{[a(\alpha^2 + \beta^2) + 2\alpha\beta b]I - [b(\alpha^2 + \beta^2) + 2\alpha\beta a]S}{(a^2 - b^2)(\alpha^2 - \beta^2)^2}.$$

Consequently, Cauchy problem (2.7) - (2.8) takes the following form

$$\begin{cases} z'(x) = \lambda \dfrac{[a(\alpha^2+\beta^2)+2\alpha\beta b]}{(a^2-b^2)(\alpha^2-\beta^2)} z(x), \\ z(0) = 0; \end{cases}$$

It is obvious that this Cauchy problem has only trivial solution at any value of λ.

Remark 2.2. This proof prompted us that $\alpha^2 - \beta^2 \neq 0$. The fact is that we did not check the conditions

1) $(a^2-b^2)(\alpha+\beta) \neq 0$,
2) $b(\alpha^2+\beta^2)+2\alpha\beta a = 0$

on compatibility.

From the second condition when $\beta = \alpha$, we have

$$b \cdot 2\alpha^2 + 2\alpha^2 \cdot a = 0, 2\alpha^2(b+a) = 0.$$

If $\alpha = 0$, then $\beta = 0$, which is not acceptable therefore $b + a = 0$, and this contradicts the first condition, therefore в $\neq \alpha$ or $\beta - \alpha \neq 0$. And the final look of the Volterra is

1) $(a^2-b^2)(\alpha^2-\beta^2) \neq 0$,
2) $b(\alpha^2+\beta^2)+2\alpha\beta a = 0$.

2.4. On basis property.

Lemma 2.5. If

a) $(a^2-b^2)(\alpha^2-\beta^2) \neq 0$,

б) function $y(x, \pm\lambda)$ is eigenfunction for the boundary value problem

$$ay'(x) + by'(1-x) = \pm\lambda y(x), x \in (0,1), \quad (2.9)$$

$$\alpha y(0) + \beta y(1) = 0, \quad (2.10)$$

where $|\alpha| + |\beta| \neq 0$, then it is eigenfunction for the Sturm - Liouville boundary value problem

$$y''(x) = \mu^2 y(x), x \in (0,1), \quad (2.11)$$

$$\begin{cases} \alpha y(0) + \beta y(1) = 0, \\ (a\alpha + b\beta)y'(0) + (a\beta + b\alpha)y'(1) = 0, \end{cases} \quad (2.12)$$

where

$$\mu^2 = \frac{\lambda^2}{b^2-a^2}. \quad (2.13)$$

Inversely, if the quantity $\mu^2 = \frac{\lambda^2}{b^2-a^2}$ is eigenvalue of the Sturm - Liouville boundary value problem

$$y''(x) = \mu^2 y(x), x \in (0,1), \quad (2.11)$$

$$\begin{cases} \alpha y(0) + \beta y(1) = 0, \\ (a\alpha + b\beta)y'(0) + (a\beta + b\alpha)y'(1) = 0, \end{cases} \quad (2.12)$$

Then the function $y(x)$ is eigenfunction for the boundary value problem

$$ay'(x) + by'(1-x) = \pm\lambda y(x), \quad (2.9)$$

$$\alpha y(0) + \beta y(1) = 0, \quad (2.10)$$

Proof. Suppose that (2.9) - (2.10) holds. Then, assuming

$$Tu(x) = au(x) + bu(1-x) = (aI + bS)u(x),$$
$$Su(x) = u(1-x)$$

we have

$$Ty'(x) = \pm\lambda y(x),$$

$$y'(x) = \pm\lambda T^{-1}y(x) = \pm\lambda\frac{aI - bS}{a^2 - b^2}y(x),$$

$$y''(x) = \pm\lambda\frac{ay'(x) + by'(1-x)}{a^2 - b^2} = \pm\lambda\frac{aI + bS}{a^2 - b^2}y'(x) =$$

$$= \frac{\pm\lambda}{a^2 - b^2}Ty'(x) = \frac{\lambda^2}{a^2 - b^2}y(x),$$

consequently,

$$-y''(x) = \frac{\lambda^2}{b^2 - a^2}y(x).$$

Further, from the equation (2.9) due to the boundary condition (2.10), we obtain

$$\alpha[ay'(0) + by'(1)] + \beta[ay'(1) + by'(0)] = \pm\lambda[\alpha y(0) + \beta y(1)] = 0.$$

Therefore, the second boundary condition has the form

$$(a\alpha + b\beta)y'(0) + (a\beta + b\alpha)y'(1) = 0.$$

Inversely, let function $y(x,\mu)$ be a eigenfunction of the Sturm – Liouville boundary value problem

$$y''(x) = \mu^2 y(x), x \in (0,1), \tag{2.11}$$

$$\begin{cases} \alpha y(0) + \beta y(1) = 0, \\ (a\alpha + b\beta)y'(0) + (a\beta + b\alpha)y'(1) = 0, \end{cases} \tag{2.12}$$

where

$$\mu^2 = \frac{\lambda^2}{b^2 - a^2}. \tag{2.13}$$

We find a eigenfunction, corresponding to the eigenvalue μ^2.
General solution of the equation (2.11) has the form

$$y(x,\mu) = Ae^{i\mu x} + Be^{-i\mu x},$$

where A, B are arbitrary constants. Putting this expansion into the boundary condition (2.12), we receive a system of homogeneous equations for unknowns A, B.

$$\alpha y(0) + \beta y(1) = \alpha(A + B) + \beta(Ae^{i\mu} + Be^{-i\mu}) =$$

$$= A(\alpha + \beta e^{i\mu}) + B(\alpha + \beta e^{-i\mu}) = 0;$$

$$y'(x,\mu) = i\mu(Ae^{i\mu x} - Be^{-i\mu x}), y'(0) = i\mu(A - B),$$

$$y'(1) = i\mu(Ae^{i\mu} - Be^{-i\mu}),$$

$$(a\alpha + b\beta)i\mu(A - B) + (a\beta + b\alpha)i\mu(Ae^{i\mu} - Be^{-i\mu}) = 0, i\mu \neq 0;$$

$$(A - B)(a\alpha + b\beta) + (a\beta + b\alpha)(Ae^{i\mu} - Be^{-i\mu}) = 0,$$

$$A[a\alpha + b\beta + (a\beta + b\alpha)e^{i\mu}] + B[-a\alpha - b\beta - (a\beta + b\alpha)e^{-i\mu}] = 0,$$

$$\begin{cases} A(\alpha + \beta e^{i\mu}) + B(\alpha + \beta e^{-i\mu}) = 0, \\ A[a\alpha + b\beta + (a\beta + b\alpha)e^{i\mu}] + B[-a\alpha - b\beta - (a\beta + b\alpha)e^{-i\mu}] = 0. \end{cases}$$

We calculate determinant of this system of equations

$$\Delta(\mu) = \begin{vmatrix} \alpha + \beta e^{i\mu} & \alpha + \beta e^{-i\mu} \\ a\alpha + b\beta + (a\beta + b\alpha)e^{i\mu} & -a\alpha - b\beta - (a\beta + b\alpha)e^{-i\mu} \end{vmatrix} =$$

$$= -(\alpha + \beta e^{i\mu})[a\alpha + b\beta + (a\beta + b\alpha)e^{-i\mu}] -$$

$$-(\alpha + \beta e^{-i\mu})[a\alpha + b\beta + (a\beta + b\alpha)e^{i\mu}] = 0. \quad (2.14)$$

Transforming this expression, we get

$$\Delta(\mu) = -[\alpha(a\alpha + b\beta) + \alpha(a\beta + b\alpha)e^{-i\mu} + \beta(a\alpha + b\beta)e^{i\mu} + \beta(a\beta + b\alpha) +$$

$$+\alpha(a\alpha + b\beta) + \alpha(a\beta + b\alpha)e^{i\mu} + \beta(a\alpha + b\beta)e^{-i\mu} + \beta(a\beta + b\alpha)] =$$

$$= -\{2\alpha(a\alpha + b\beta) + 2\beta(a\beta + b\alpha) + [\alpha(a\beta + b\alpha) + \beta(a\alpha + b\beta)]e^{-i\mu} +$$

$$+[\alpha(a\beta + b\alpha) + \beta(a\alpha + b\beta)]e^{i\mu}\} =$$

$$= -\{2\alpha^2 a + 2\alpha\beta b + 2\beta^2 a + 2\alpha\beta b +$$

$$+[\alpha(a\beta + b\alpha) + \beta(a\alpha + b\beta)](e^{i\mu} + e^{-i\mu})\} =$$

$$= -2[(\alpha^2 + \beta^2)a + 2\alpha\beta b + (\alpha\beta a + b\alpha^2 + \beta\alpha a + b\beta^2)\cos\mu] =$$

$$= -2\{a(\alpha^2 + \beta^2) + 2\alpha\beta b + [b(\alpha^2 + \beta^2) + 2\alpha\beta a]\cos\mu\}.$$

Remark 2.3. If

1) $b(\alpha^2 + \beta^2) + 2\alpha\beta a = 0$;
2) $a(\alpha^2 + \beta^2) + 2\alpha\beta b \neq 0$

then $\Delta(\mu) \neq 0$, i.e. the problem is Volterra.

In this case:

$$2) - 1) = (б^2 + \beta^2)(a - b) + 2\alpha\beta(b - a) = (a - b)(\alpha - \beta)^2 \neq 0,$$

$$2) + 1) = (\alpha^2 + \beta^2)(a + b) + 2\alpha\beta(b + a) = (a + b)(\alpha + \beta)^2 \neq 0.$$

Therefore, $(a^2 - b^2)(\alpha^2 - \beta^2)^2 \neq 0$, that coincides with the first condition of Volterra property, see (2.2).

Supposing $b(б^2 + \beta^2) + 2\alpha\beta a \neq 0$, from the equation $\Delta(\mu) = 0$ we have

$$\cos\mu = -\frac{a(\alpha^2 + \beta^2) + 2\alpha\beta b}{b(\alpha^2 + \beta^2) + 2\alpha\beta a},$$

$$\mu = \pm arccos\left[-\frac{a(\alpha^2 + \beta^2) + 2\alpha\beta b}{b(\alpha^2 + \beta^2) + 2\alpha\beta a}\right] + 2n\pi, n = 0, \pm1, \pm2, \ldots$$

We investigate multiplicity of eigenvalues

$$\dot{\Delta}(\mu) = 2[b(\alpha^2 + \beta^2) + 2\alpha\beta a]\sin\mu.$$

If $\dot{\Delta}(\mu) = 0$, then $\sin\mu = 0, \cos\mu = \pm1$. Thus

$$-\frac{a(\alpha^2 + \beta^2) + 2\alpha\beta b}{b(\alpha^2 + \beta^2) + 2\alpha\beta a} = \pm1.$$

a) If

$a(\alpha^2 + \beta^2) + 2\alpha\beta b = b(\alpha^2 + \beta^2) + 2\alpha\beta a$, then

$$(a - b)(\alpha^2 + \beta^2) + 2\alpha\beta(b - a) = (a - b)(\alpha^2 + \beta^2 - 2\alpha\beta) =$$

$$= (a - b)(\alpha - \beta)^2 = 0;$$

б) If

$$a(\alpha^2+\beta^2)+2\alpha\beta b=-b(\alpha^2+\beta^2)-2\alpha\beta a,$$

$$(\alpha^2+\beta^2)(a+b)+2\alpha\beta(b+a)=(a+b)(\alpha^2+\beta^2+2\alpha\beta)=$$

$$=(a+b)(\alpha+\beta)^2=0.$$

In oue case, $(a^2-b^2)(\alpha^2-\beta^2)\neq 0$, therefore $\dot{\Delta}(\mu)\neq 0$ and the Sturm - Liouville problem does not have associated functions.

Supposing,

$$A=K(\alpha+\beta e^{-i\mu}), B=(-\alpha-\beta e^{i\mu})K,$$

where K is arbitrary constant, we obtain (find) eigenfunctions of the Sturm – Liouville boundary value problem

$$y(x,\mu)=K(\alpha+\beta e^{-i\mu})e^{i\mu x}-K(\alpha+\beta e^{i\mu})e^{-i\mu x}.$$

Then

$$y'(x,\mu)=K[i\mu(\alpha+\beta e^{-i\mu})e^{i\mu x}+i\mu(\alpha+\beta e^{i\mu})e^{-i\mu x}],$$

$$y'(1-x,\mu)=K[i\mu(\alpha+\beta e^{-i\mu})e^{i\mu(1-x)}+i\mu(\alpha+\beta e^{i\mu})e^{-i\text{м}(1-x)}]=$$

$$=K[i\mu(\alpha e^{i\mu}+\beta)e^{-i\mu x}+i\mu(\alpha e^{-i\mu}+\beta)e^{i\mu x}];$$

$$ay'+by'(1-x)=i\mu K\{[a(\alpha+\beta e^{-i\mu})+b(\alpha e^{-i\mu}+\beta)]e^{i\mu x}+$$

$$+[a(\alpha+\beta e^{i\mu})+b(\alpha e^{i\mu}+\beta)]e^{-i\mu x}\}.$$

Further, from the equality $\Delta(\text{м})=0$ it follows that rows of this determinant (2.14) are linear dependent, therefore we have

$$a\alpha+a\beta e^{-i\mu}+b\alpha e^{-i\mu}+b\beta=$$

$$=a\alpha+b\beta+(a\beta+b\alpha)e^{-i\mu}=-C(\alpha+\beta e^{-i\mu}), \qquad (2.15)$$

$$a\alpha+a\beta e^{i\mu}+b\alpha e^{i\mu}+b\beta=$$

$$=a\alpha+b\beta+(a\text{в}+b\alpha)e^{i\mu}=C(\alpha+\beta e^{i\mu}), \qquad (2.16)$$

Consequently,

$$ay'+by'(1-x)=i\mu K\cdot C[-(\alpha+\beta e^{-i\mu})e^{i\mu x}+(\alpha+\beta e^{i\mu})e^{-i\mu x}]=$$

$$=i\mu K\cdot C[(\alpha+\beta e^{-i\mu})e^{i\mu x}-(\alpha+\beta e^{i\mu})e^{-i\mu x}]=$$

$$=-i\mu\cdot Cy(x,\mu)=-\frac{\lambda}{\sqrt{a^2-b^2}}Cy(x,\mu). \qquad (2.17)$$

Calculate the constant C. If we sum up formulas (2.15) and (2.16), then we get

$$2(a\alpha+b\beta)+(a\beta+b\alpha)(e^{i\mu}+e^{-i\mu})=C(e^{i\mu}-e^{-i\mu})\cdot\beta,=>$$

$$a\alpha+b\beta+(a\beta+b\alpha)\cos\mu=C\cdot\beta i\sin\mu.$$

Subtracting the formula (2.15) from (2.16), we have

$$(a\beta+b\alpha)(e^{i\mu}-e^{-i\mu})=C[2\alpha+\beta(e^{i\mu}+e^{-i\mu})],=>$$

$$i(a\beta+b\alpha)\sin\mu=C[\alpha+\beta\cos\mu];$$

Consequently,

$$C=\frac{i(a\beta+b\alpha)\sin\mu}{\alpha+\beta\cos\mu}=\frac{a\alpha+b\beta+(a\beta+b\alpha)\cos\mu}{i\beta\sin\mu};$$

In our situation

$$\cos\mu=-\frac{a(\alpha^2+\beta^2)+2\alpha\beta b}{b(\alpha^2+\beta^2)+2\alpha\beta a},$$

Thus

$$\alpha+\beta\cos\mu=\alpha-\frac{a\beta(\alpha^2+\beta^2)+2\alpha\beta^2 b}{b(\alpha^2+\beta^2)+2\alpha\beta a}=$$

$$=\frac{\alpha b(\alpha^2+\beta^2)+2\alpha^2\beta a-a\beta(\alpha^2+\beta^2)-2\alpha\beta^2 b}{b(\alpha^2+\beta^2)+2\alpha\beta a}=$$

$$=\frac{(\alpha^2+\beta^2)(\alpha b-a\beta)+2\alpha\beta(\alpha a-\beta b)}{b(\alpha^2+\beta^2)+2\alpha\beta a}=$$

$$=\frac{b[\alpha(\alpha^2+\beta^2)-2\alpha\beta^2]+a[-\beta(\alpha^2+\beta^2)+2\alpha^2\beta]}{b(\alpha^2+\beta^2)+2\alpha\beta a}=$$

$$=\frac{b\alpha(\alpha^2-\beta^2)+a\beta(\alpha^2-\beta^2)}{b(\alpha^2+\beta^2)+2\alpha\beta a}=\frac{(\alpha^2-\beta^2)(b\alpha+a\beta)}{b(\alpha^2+\beta^2)+2\alpha\beta a};$$

$$\sin\mu=\pm\sqrt{1-\frac{[a(\alpha^2+\beta^2)+2\alpha\beta b]^2}{[b(\alpha^2+\beta^2)+2\alpha\beta a]^2}}=\frac{\pm\sqrt{b^2-a^2}(\alpha^2-\beta^2)}{b(\alpha^2+\beta^2)+2\alpha\beta a};$$

$$C=\frac{\pm i(a\beta+b\alpha)\sqrt{b^2-a^2}(\alpha^2-\beta^2)}{b(\alpha^2+\beta^2)+2\alpha\beta a}:\frac{(б^2-\beta^2)(b\alpha+a\beta)}{b(\alpha^2+\beta^2)+2\alpha\beta a}=$$

$$=\pm i\sqrt{b^2-a^2}=\pm\sqrt{a^2-b^2}.$$

Putting this expression into the formula (2.17), we have

$$ay'(x)+by'(1-x)=\mp\lambda y(x,\mu),$$

which was required to prove.

Problem. Prove that the system of eigenfunctions of the Sturm - Liouville boundary value problem is basis

$$-y''(x)=\mu^2 y(x), x\in(0,1), \tag{2.11}$$

$$\begin{cases} \alpha y(0)+\beta y(1)=0, \\ (\alpha a+b\beta)y'(0)+(a\beta+b\alpha)y'(1)=0, \end{cases} \tag{2.12}$$

where $(\alpha^2-\beta^2)(a^2-b^2)\neq 0$, by the Kesselman - Mikhailov Test [57-58].

Solution. Boundary matrix of the boundary value problem (2.11) - (2.12) has the form

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \alpha a+b\beta & 0 & a\beta+b\alpha \end{pmatrix}.$$

We calculate minors of this matrix

$$\Delta_{12}=\alpha(\alpha a+b\beta), \Delta_{13}=0, \Delta_{14}=\alpha(a\beta+b\alpha),$$

$$\Delta_{23}=-\beta(\alpha a+\beta b), \Delta_{24}=0, \Delta_{34}=\beta(a\beta+b\alpha);$$

Hence,

$$\Delta_{14}+\Delta_{32}=\alpha(a\beta+b\alpha)+\beta(\alpha a+\beta b)=$$

$$=b(\alpha^2+\beta^2)+2\alpha\beta a\neq 0,$$

otherwise the problem is Volterra.

First we check the Birkhoff regularity condition [54]; for this, we rearrange rows of the boundary matrix

$$\begin{pmatrix} a_0 & a_1 & b_0 & b_1 \\ 0 & \alpha a + b\beta & 0 & a\beta + b\alpha \\ \alpha & 0 & \beta & 0 \\ c_0 & c_1 & d_0 & d_1 \end{pmatrix}$$

1) $a_1 d_1 - b_1 c_1 = 0,$
2) $a_1 d_1 - b_1 c_1 = 0, |a_1| + |b_1| = |\alpha a + b\beta| + |a\beta + b\alpha| > 0,$

$$b_1 c_0 + a_1 d_0 = \alpha(a\text{в} + b\alpha) + \beta(\alpha a + \beta b) = b(\alpha^2 + \beta^2) + 2\alpha\beta a \neq 0,$$

3) $a_1 = b_1 = c_1 = d_1 = 0, a_0 d_0 - b_0 c_0 \neq 0.$

It is obvious that in our case conditions 1) and 3) coincide, only condition 2) remains.

If $|a_1| + |b_1| = 0$, then $a_1 = 0$ and $b_1 = 0$. Then we get

$$\begin{cases} \alpha a + b\beta = 0, \\ a\beta + b\alpha = 0. \end{cases}$$

Since $|a| + |b| \neq 0$, then $\alpha^2 - \beta^2 = 0$; similarly from the condition $|\alpha| + |\beta| \neq 0$ we have that $a^2 - b^2 = 0$, which is not possible by our condition. Consequently, our boundary value problems (2.12) are regular by the second part of the Birkhoff condition [56].

The Kesselman condition [57] is

$$\Delta_{14}^2 + \Delta_{32}^2 - \Delta_{12}^2 - \Delta_{34}^2 \neq 0.$$

In our case,

$$\Delta_{14}^2 + \Delta_{32}^2 = \alpha^2(a\beta + b\alpha)^2 + \beta^2(\alpha a + \beta b)^2,$$

$$\Delta_{12}^2 = \alpha^2(\alpha a + \beta b)^2, \Delta_{34}^2 = \beta^2(a\beta + b\alpha)^2, =>$$

$$\Delta_{14}^2 + \Delta_{32}^2 - \Delta_{12}^2 - \Delta_{34}^2 = (\alpha^2 - \beta^2)(a\beta + b\alpha)^2 + (\beta^2 - \alpha^2(\alpha a + \beta b)^2) =$$

$$= (\alpha^2 - \beta^2)[(a\beta + b\alpha)^2 - (\alpha a + \beta b)^2] =$$

$$= (\alpha^2 - \beta^2)[(a\beta + b\alpha - \alpha a - \beta b)(a\beta + b\alpha + \alpha a + \beta b)] =$$

$$= (\alpha^2 - \beta^2)[a(\beta - \alpha) + b(\alpha - \beta)][\text{б}(a + b) + \beta(a + b)] =$$

$$= (\alpha^2 - \beta^2)(\alpha - \beta)(b - a)(a + b)(\alpha + \beta) =$$

$$= (\alpha^2 - \beta^2)(\alpha^2 - \beta^2)(b^2 - a^2) = (\alpha^2 - \beta^2)^2(b^2 - a^2) \neq 0.$$

Consequently, the eigenfunctions of the Sturm – Liouville boundary value problem (2.11) - (2.12) form a Riesz basis in the space $L^2(0,1)$.

Theorem 2.2. If

a) $(\alpha^2 - \beta^2)(a^2 - b^2) \neq 0,$

b) $b(\alpha^2 + \beta^2) + 2\alpha\beta a \neq 0,$

then eigenfunctions of the bpundary value probelm

$$ay'(x) + by'(1 - x) = \lambda y(x), x \in (0,1), \quad (1.1)$$

$$\alpha y(0) + \beta y(1) = 0, \quad (1.2)$$

form a Riesz basis in the space $L^2(0,1)$.

Remark 2.5.

If $a = 0, b = 1$, then the boundary value problem (1.1)-(1.2) takes the form

$$y'(1 - x) = \lambda y(x), x \in (0,1),$$

$$\alpha y(0) + \beta y(1) = 0.$$

Thus the conditions a) and b) of Theorem 2.2. are transformed as follows:

a)$'$ $\alpha^2 - \beta^2 \neq 0,$

b)$'$ $\alpha^2 + \beta^2 \neq 0$

or $(\alpha^2 - \beta^2)(\alpha^2 + \beta^2) \neq 0$, or $\alpha^4 - \beta^4 \neq 0$, that coincides with the results of [28].

3. Research Results

We consider in the space $L^2(0,1)$ the following boundary value problem

$$Ly = ay'(x) + by'(1-x) = \lambda y(x), x \in (0,1), \quad (1.1)$$

$$\alpha y(0) + \beta y(1) = 0, \quad (1.2)$$

where a, b, α, β are arbitrary, previously known constants, and λ is a spectral parameter, $y(x)$ is a desired function from the class $C^1(0,1) \cap C[0,1]$, and we formulate the obtained results.

Theorem 3.1. The boundary value problem (1.1) - (1.2) is Volterra if and only if

1) $(a^2 - b^2)(\alpha^2 - \beta^2) \neq 0$; (2.2)
2) $b(\alpha^2 + \beta^2) + 2\alpha\beta a = 0$. (2.3)

Theorem 3.2. If

а) $(\alpha^2 - \beta^2)(a^2 - b^2) \neq 0,$
б) $b(\alpha^2 + \beta^2) + 2\alpha\beta a \neq 0;$

then eigenfunctions of the boundary value problem

$$ay'(x) + by'(1-x) = \lambda y(x), x \in (0,1), \quad (1.1)$$

$$\alpha y(0) + \beta y(1) = 0, \quad (1.2)$$

form a Riesz basis in the space $L^2(0,1)$.

4. Discussion of Results.

Remark 4.1. If

$$\begin{cases} b(\alpha^2 + \beta^2) + 2\alpha\beta a = 0, \\ b2\alpha\beta + (\alpha^2 + \beta^2)a = 0, \end{cases}$$

then $(\alpha^2 + \beta^2)^2 - 4\alpha^2\beta^2 = 0$, i.e.

$$(\alpha^2 + \beta^2)^2 - 4\alpha^2\beta^2 = (\alpha^2 + \beta^2 - 2\alpha\beta)(\alpha^2 + \beta^2 + 2\alpha\beta) = (\alpha - \beta)^2(\alpha + \beta)^2$$
$$= (\alpha^2 - \beta^2)^2 = 0.$$

Remark 4.2.

a) If $C = -\sqrt{a^2 - b^2}.$, then from the formula (2.15) we have

$$a\alpha + b\beta + (a\beta + b\alpha)e^{-i\mu} = \sqrt{a^2 - b^2}.\left(\alpha + \beta e^{-i\mu}\right),$$

$$a\alpha + b\beta + (a\beta + b\alpha)e^{-i\mu} = \beta\sqrt{a^2 - b^2}e^{-i\mu} + \alpha\sqrt{a^2 - b^2},$$

$$\left(a\beta + b\alpha - \beta\sqrt{a^2 - b^2}\right)e^{-i\mu} = \alpha\sqrt{a^2 - b^2} - a\alpha - b\beta,$$

$$e^{-i\mu} = \frac{б\sqrt{a^2 - b^2} - a\alpha - b\beta}{a\beta + b\alpha - \beta\sqrt{a^2 - b^2}},$$

and from the formula (2.16) we have

$$e^{i\mu} = \frac{-\alpha\sqrt{a^2-b^2} - a\alpha - b\beta}{a\beta + b\alpha + \beta\sqrt{a^2-b^2}}, =>$$

consequently, $e^{-i\mu} \cdot e^{i\mu} = 1$.

б) If $C = \sqrt{a^2-b^2}$., then from the formula (2.15) we have

$$a\alpha + b\beta + (a\beta + b\alpha)e^{-i\mu} = -\sqrt{a^2-b^2}.\left(\alpha + \beta e^{-i\mu}\right),$$

$$a\alpha + b\beta + (a\beta + b\alpha)e^{-i\mu} = -\beta\sqrt{a^2-b^2}e^{-i\mu} - \alpha\sqrt{a^2-b^2},$$

$$\left(a\beta + b\alpha + \beta\sqrt{a^2-b^2}\right)e^{-i\mu} = -\alpha\sqrt{a^2-b^2} - a\alpha - b\beta,$$

$$e^{-i\mu} = -\frac{\alpha\sqrt{a^2-b^2} + a\alpha + b\beta}{a\beta + b\alpha + \beta\sqrt{a^2-b^2}}.$$

5. Findings. Operator corresponding to this boundary value problem is not semi-bounded; therefore, variational methods are not suitable to study such problems, and this is a distinctive feature of this problem. In our opinion, such operators can be used to construct non-local transformation operators, and apply them to study spectral properties of operator sheaves.

ӘОЖ 517.929

А.Ш.Шалданбаев[1], С.М.Шаленова[2], М.Б.Иванова[3], А.А.Шалданбаева[4]

[1]Халықаралық Silkway университеті, Шымкент қ., Казақстан;
[2]Ш. Есенов атындағы Каспий мемлекеттік технологиялар мен инжиниринг университеті, Ақтау қ., Қазақстан;
[3]Оңтүстік Қазақстан медициналық академиясы, Шымкент қ., Казақстан;
[4]Аймақтық әлеуметтік-инновациялық университеті, Шымкент қ., Казақстан;

АРГУМЕНТІ АУЫТҚЫҒАН БІРІНШІ РЕТТІ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУДІҢ ШЕКАРАЛЫҚ ЕСЕБІНІҢ СПЕКТРӘЛДІК ҚАСИЕТТЕРІ ТУРАЛЫ

Аннотация. Бұл еңбекте Аргументі ауытқыған бірінші ретті дифференциалдық теңдеудің шекаралық есебінің спектрәлдік қасиеттері зерттелді. Бір айта кетері, ауытқу теңдеудің үлкен ретті туындыларында кездеседі және оны қосымша шарттарсыз төменгі ретті мүшелерге ауыстыруға болмайды. Спектрәлді қасиеттер ретінде, біз оның меншікті векторлары мен оларға еншілес векторлар системасының толымдылығы мен базистігін таныймыз, сонымен бірге осы қасиеттер қатарына оның вөлтерлігін де жатқызамыз.

Түйін сөздер: аргументі ауытқыған теңдеу, толымдылық, базистік, вөлтерлік, Гаалдың формуласы, Лидскийдің теоремасы, Штурм –Лиувиллдің операторы, Рисстің базисі.

УДК 517.929

А.Ш.Шалданбаев[1], С.М.Шаленова[2], М.Б.Иванова[3], А.А.Шалданбаева[4]

[1]Международный университет Silkway, г. Шымкент, Казахстан;
[2]Каспийский государственный университет технологий и инжиниринга им. Ш.Есенова, г.Актау, Казахстан;
[3]Южно-Казахстанская медицинская академия, г. Шымкент, Казахстан;
[4]Региональный социально-инновационный университет, г. Шымкент, Казахстан.

О СПЕКТРАЛЬНЫХ СВОЙСТВАХ КРАЕВОЙ ЗАДАЧИ УРАВНЕНИЯ ПЕРВОГО ПОРЯДКА С ОТКЛОНЯЮЩИМСЯ АРГУМЕНТОМ

Аннотация. В данной работе изучены спектральные свойства краевой задачи дифференциального уравнения первого порядка с постоянными коэффициентами и отклоняющимся аргументом, при этом отклонение присутствует при старшем члене уравнения, и его нельзя перебрасывать без дополнительного условия на младшие члены уравнения. Под спектральными свойствами мы имеем в виду полноту и базисность системы собственных и присоединенных функций краевой задачи, а также вольтерровость.

Ключевые слова: уравнение с отклоняющимся аргументом, полнота, базисность, вольтерровость, формула Гаала, теорема Лидского, оператор Штурма - Лиувилля, базис Рисса.

Information about authors:

Shaldanbayev A.Sh. – doctor of physico-mathematical Sciences, associate Professor, head of the center for mathematical modeling, «Silkway» International University, Shymkent; http://orcid.org/0000-0002-7577-8402;

Shalenova S.M. - Yessenov University; Aktau; https://orcid.org/0000-0002-1378-3310;

Ivanova M.B. - South Kazakhstan medical Academy, Shymkent; https://orcid.org/0000-0001-6548-0699;

Shaldanbayeva A.A. - "Regional Social-Innovative University", Shymkent; https://orcid.org/0000-0003-2667-3097

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