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UNIVERSAL POSITIVE PREORDERS

Abstract. In this paper, we investigate universal objects in the class of positive preorders with respect to computable reducibility, we constructed a computable numbering of this class and proved theorems on the existence of a universal positive lattice and a universal weakly precomplete.

Keywords. Computable reducibility on preorders, universal positive preorder, computable numbering, positive lattice, positive linear order, weakly pre-complete positive preorder.

The class of positive (computably enumerable) equivalences, which is a proper subclass of the class of positive preorders, first appeared in the paper of Yu.L. Ershov [1]. In recent decades, interest has increased in a research of positive equivalences and positive preorders with respect to natural computable reducibility(see, for example, [2], [3], [4] and [5]). In the class of positive equivalences, universal objects are well-described (see [3], [4]), while universal preorders are poorly known, though attract huge interest, both from the point of view of the computability theory, and for applications in theoretical computer science. Our work is devoted to studying of positive preorders defined on the set of natural numbers ω with respect to computable reducibility, defined as follows: positive preorder P is computably reduced to positive preorder Q (symbolically, $P \leq_c Q$), if for any $x, y \in \omega$ there exists a computable function f such that $x\ P\ y$ if and only if $f(x)\ Q\ f(y)$. A positive preorder is universal if any positive preorder reduces to it. The first references to universal preorders meet in the work of Italian mathematicians F. Montagny and A. Sorbi, [6].

We follow the standard notation from the book [7]: Post's numbering of computably enumerable (c.e.) sets is denoted by$\{W_x\}_{x\in\omega}$, φ_e denotes the partially computable function of the Kleene's number$e$, and the standard coding of pairs of natural numbers is denoted by $\langle\cdot,\cdot\rangle$. Through $l(\cdot)$ and $r(\cdot)$ we denote computable functions that, by the code of the pair, restore its left and right components.

We briefly recall the basic concepts and introduce some notations used below in the paper.

Let S be some at most countable set, then an arbitrary map of the natural numbers ω to the set S is called the numbering of the family S. The numbering of the family S of c.e. sets are called computable if the set $\{\langle x, y\rangle : x \in \nu(y)\}$is c.e. set.

A positive preorder P is called universal in the class of preorders K if $P \in K$ and $Q \leq_c P$ for any $Q \in K$. For the preorder P, we denote by $[x]_P$ the equivalence class of x with respect to P, i.e.$[x]_P = \{y : x\ P\ y\ \&\ y\ P\ x\}$.

By Id we denote the identical preorder$\{(x, x) : x \in \omega\}$, and by $\mathcal{P}_1$ we denote the family of all positive preorders.

If$\leq$ is a partial ordering on some set M and x, y are elements of this set, then $z \in M$ is called the least upper bound of the elements $x, y(\sup(x, y))$, if $x, y \leq z$ and $\forall t\ [x, y \leq t \rightarrow z \leq t]$, and $z \in M$ is called the greatest lower bound of the elements $x, y(\inf(x, y))$, if $z \leq x, y$ and $\forall t\ [t \leq x, y \rightarrow t \leq z]$. It is clear that $x \leq y$ if and only if $\sup(x, y) = y$ and$\inf(x, y) = x$.

Recall that a lattice is a partially ordered set, where any two elements have the smallest upper bound and greatest lower bound.

If A is a set of natural numbers and n is a number, then $A \upharpoonright n$ denotes the set $A \cap \{0,1,\dots,n\}$.

Proposition 1. There is a computable numbering α of the family$\mathcal{P}_1$.

Proof. Fix some computable approximation $\{W_x^s\}_{s\in\omega}$ of the setW_x. We will construct a computable approximation $\{\alpha^s\}_{s\in\omega}$ of the numbering α and the computable function $b(x,s)$as follows:

CONSTRUCTION

Stage 0.For any $x \in \omega$we set$\alpha^0(x) = Id$ and $b(x,0) = n$for some$n > 1$.Go to the nextstage.

Stage $s+1$.For all$x \le s$do the following: if$W_x^s \upharpoonright b(x,s)$is a preorder, then we assume$\alpha^{s+1}(x) = \alpha^s(x) \cup W_x^s \upharpoonright b(x,s)$and $b(x,s+1) = b(x,s)+1$.We call this stage an “expanding stage” for $\alpha\,(x)$. Go to the end of the stage.

End of stage.For all$x,s \in \omega$we assume that

- $\alpha^{s+1}(x) = \alpha^s(x)$,if$\alpha^{s+1}(x)$is undefined;
- $b(x,s+1) = b(x,s)$,if$b(x,s+1)$is undefined.

Go to the next stage.

Lemma 1. If for $\alpha(x)$ there are infinitely many “expanding stages”, then $\alpha(x) = W_x$.

Theproofisobvious.

Lemma2.$\alpha(x)$is a positive preorder for any $x \in \omega$.

Proof. Obviously, $\alpha^s(x)$is a positive preorder at each stage s.There are two cases: the first, when for $\alpha(x)$there are infinitely many “extending stages”.For this case, by Lemma 1, $\alpha(x) = W_x$. This can happen only when $W_x$is a positive preorder. And if the “expanding stages” are the finite number, then by construction

$$\alpha(x) = \lim_s \alpha^s(x) = \alpha^{s'}(x),$$

where$s'$is the last “expanding stage”. It follows from the reasoning above that $\alpha^{s'}(x)$is a positive preorder.

Lemma 3. The numbering α is a computable numbering of the family $\mathcal{P}_1$

Proof. First, we prove that α is a numbering of the family$\mathcal{P}_1$.Let $P \in \mathcal{P}_1$be an arbitrary positive preorder. Since P is c.e. set, there exists x such that $P = W_x$. Since W_x is a preorder, then for $\alpha(x)$there are infinitely many “extending stages”.Therefore, by Lemma 1, $\alpha(x) = W_x = P$. Moreover, the fact that α is a computable numbering follows from the construction efficiency.

Corollary 1. The numbering α constructed in Proposition 1 is universal in the class $Com(\mathcal{P}_1)$of computable numberings of the family $\mathcal{P}_1$.

Proof. Let β be an arbitrary computable numbering of the family $\mathcal{P}_1$. Since $\mathcal{P}_1$ is thefamilyof c.e. sets, then $\beta \le W_x$. Let $\beta \le W_x$via the function f. Then $\beta \le \alpha$via the function f. Indeed, for any$x\beta(x)$ is a positive preorder. Since $\beta(x) = W_{f(x)}$ and $W_{f(x)}$is a positive preorder, then $W_{f(x)} = \alpha(f(x))$.

Corollary 2. There is a universal positive preorder with respect to computable reducibility $\le_c$.

Proof. We construct the preorder U as follows:

$$x\ U\ y \Leftrightarrow l(x) = l(y) \,\&\, r(x)\ \alpha(l(x))\ r(y)$$

We show that any positive preorder P is computably reduced to U. Since P is a positive preorder, then there exists e such that $\alpha(e) = P$.Consequently,$P \le_c U$by the function $f(x) = \langle e,x\rangle$. ■

Consider special types of positive preorders:

We say that a positive preorder P is a positive linear order if the factor set $\omega/ER(P)$with the preorder $\le_P$given by the rule:

$$[x]_{ER(P)} \le_P [y]_{ER(P)} \Leftrightarrow \exists x' \exists y' (x' \in [x]_{ER(P)} \& y' \in [y]_{ER(P)} \& x' P\, y')$$

is linearly ordered.

Definition. A positive prelattice is called a positive preorder P whose factor structure $(\omega/ER(P), sup, inf, \leq_P)$ is a lattice and the functions sup andinf are partially computable. Here by $ER(P)$ denotes the greatest equivalence, which is contained in P, i.e. $ER(P) = \{(x,y): xPy \,\&\, yPx\}$.

Positive linear preorders, which are a frequent case of positive prelattices, are defined similarly. We say that a positive preorder P is a positive linear preorder if the factor set $\omega/ER(P)$ with the preorder $\leq_P$ given by the rule:

$$[x]_{ER(P)} \leq_P [y]_{ER(P)} \Leftrightarrow \exists x' \exists y' \big(x' \in [x]_{ER(P)} \& y' \in [y]_{ER(P)} \& x' P\, y'\big)$$

is a linearly ordered set. The existence of a universal prelattice in the class of positive linear preorders was proved in [8].

Theorem 1. Let $\mathcal{R}$ be the family of all positive prelattices.Thereexists a family $\mathcal{T} \subseteq \mathcal{R}$ and a computable numbering of the family $\mathcal{T}$ such that for any $R \in \mathcal{R}$ there exists a prelattice $T \in \mathcal{T}$ for which $R \leq_c T$.

Proof. If R is a prelattice and $T \subseteq R$, then by $[T]$ we denote the closure of the set T with respect to sup and inf. Note that for any prelattice R and any finite set $T \subseteq R$, the closure $[T]$ is also a finite set.

Let π be the computable numbering of the family of all positive preorders and let $\{\pi^s(x)\}_{s\in\omega}$ be the computable approximation of the preorder $\pi(x)$. We construct an approximation of the computable numbering α and the family $\mathcal{T}$ as follows:

CONSTRUCTION

Stage 0. We define$\alpha^0(x) = Id$for all$x \in \omega$.The set $\{0,1\}$ is declared the effective range of the $\pi^0(x)$.Go to the next stage.

Stage$2s+1$. Consider the following cases:

Case 1. If the effective range of $\pi^s(x)$ is a prelattice, then copy the effective range to $\alpha^{s+1}(x)$, i.e.we select the fresh (has never been used up to this point) elements a_i for all i from the effective range of $\pi^s(x)$.The smallest number $n$which is outside the effective range of $\pi^s(x)$ is added to the effective range of$\pi^{s+1}(x)$. Go to the next stage.

Case 2.If the effective range of $\pi^s(x)$ is not a prelattice, then add all the supandinf elements of the effective range of $\pi^s(x)$ to the effective rangeof $\pi^{s+1}(x)$, if there are any. Go to the next stage.

Stage$2s+2$. Choose the least number $n \notin range\{a_i\}$and declare this element equivalent to the element 0. Go to the next stage.

Lemma1. For any $x \in \omega$, $\alpha(x)$ is a positive prelattice.

It's not so hard to show, since we copy only the positive prelattice to the elements a_i, and all other elements are equivalent to 0. Forarbitraryelementsx, y:

1) If$x = a_i$and$y = a_j$for some i, j, then$\sup(x,y) = \sup(a_i, a_j) = a_k$where$k = \sup(i,j)$in the effective range of$\pi(x)$.

2) If$x = a_i$and$y \notin range\{a_i\}$,then$\sup(x,y) = \sup(a_i, 0) = a_k$where$k = \sup(i, 0)$in the effective range of$\pi(x)$.

3) If$x \notin range\{a_i\}$and$y = a_j$forsome$j$, this case is similar to case 2.

4) If$x, y \notin range\{a_i\}$, then$\sup(x,y) = \inf(x,y) = 0$.

For infwe carry out a similar reasoning.

Lemma2. If $\pi(x)$ is a positive prelattice, then atstages$2s+1$, case 1 is repeated infinitely often and case 2 cannot be repeated infinitely times without case 1.

The proof follows from the remark about the finiteness of the closure of finite sets.

Lemma 3. Forany $R \in \mathcal{R}$, there exists a number x such that $R \leq_c \alpha(x)$.

Proof. Since R is a positive prelattice, then by Lemma 2 case 1 is performed infinitely often and all elements of R. will enter the effective range R. Therefore, reducibility is carried out bythe function $f(x) = a_x$. ■

Theorem 2. In the class of positive prelattices, there is a universal prelattice.

Proof. Let α be the computable numbering of the family T of positive prelattices. We construct a positive prelattice$U$ as follows:

$$x\ U\ y \Leftrightarrow l(x) < l(y) \vee \left[l(x) = l(y) \,\&\, r(x)\ \alpha\big(l(x)\big)\ r(y)\right].$$

If P is a positive prelattice, then $P \leq_c \alpha(e)$ for some e and $\alpha(e) \leq_c U$ by the function$f(x) = \langle e, x\rangle$. It remains to prove that the positive preorder U is a positive prelattice. Let$[x]_U$ and $[y]_U$be two different equivalence classes. If$l(x) = l(y)$ then the sup and inf of these classes coincide with the sup and inf in the positive prelattice $\alpha\big(l(x)\big)$. If$l(x) < l(y)$, then$\sup([x]_U, [y]_U) = [y]_Uand\inf([x]_U, [y]_U) = [x]_U$. The case when $l(x) > l(y)$ is similar to the previous case. ■

The following special type of positive preorders is weakly precomplete positive preorders.

Definition. A positive preorder P is called weakly precomplete [8], if for any total function φ_e, there exists an element $x_e$such that $\varphi_e(x_e)Px_e$.

Note that the concept of weaklyprecompleteness for positive preorders is identical to this concept for positive equivalences, which was originally introduced in [9]and found to be very useful in the study of positive equivalences (see review [4]).

Theorem 3. For any positive preorder Pthere is a weakly precomplete positive preorder Q such that $P \leq_c Q$.

Proof. Let P be an arbitrary positive preorder. We construct a positive preorder Q as follows:

CONSTRUCTION

Stage 0. Let$Q^0 = P \oplus Id$. Go to the next stage.

Stage$s+1$.Let$l(s) = e$.We work withφ_e.

Let$x_e = 2e + 1$.

1) If$\varphi_e^s(x_e)\uparrow$,then$Q^{s+1} = Q^s \cup P^{s+1} \oplus Id$.

2) If$\varphi_e^s(x_e)\downarrow$, then$Q^{s+1} = Q^s \cup P^{s+1} \oplus Id \cup \{(\varphi_e(x_e), x_e), \big(x_e, \varphi_e(x_e)\big)\}$we reflexively and transitively close. Go to the next stage.

By construction, it is easy to see that the preorder Q is positive and the computable function $f(x) = 2x$performs the reduction $P \leq_c Q$. ■

Corollary. In the class of positive preorders $\mathcal{P}_1$there exists a universal weakly precomplete preorder.

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УНИВЕРСАЛЬНЫЕ ПОЗИТИВНЫЕ ПРЕДПОРЯДКИ

Аннотация.В работе исследуются универсальные объекты в классе позитивных предпорядков относительно вычислимой сводимости, строится вычислимая нумерация этого класса, доказываются теоремы о существовании универсальной позитивной решеткии универсального слабо предполного предпорядка.

Ключевыеслова. Вычислимая сводимость на предпорядках, универсальный позитивный предпорядок, вычислимая нумерация, позитивная решетка, позитивный линейный порядок, слабо предполный позитивный предпорядок.

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УНИВЕРСАЛ ПОЗИТИВ ЖАРТЫ РЕТТЕР

Аннотация. Жұмыста позитив жарты реттер класында есептелімділік көшірулер бойынша универсал объектілері зерттеледі, бұл кластың есептелімді нөмірлеуі құрылады, универсал позитив торлар және универсал жартылай толық жарты реттер табылатындығы туралы теорема дәлелденеді.

Ключевые слова. Жарты реттердегі есептелімді көшіру, универсал позитив жарты рет, есептелімді нөмірлеу, позитив тор, позитив сызықтық рет, жартылай толық позитив жарты рет.

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