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ASYMPTOTIC EXPANSION OF SOLUTION OF GENERAL BVP WITH INITIAL JUMPS FOR HIGHER-ORDER SINGULARLY PERTURBED INTEGRO-DIFFERENTIAL EQUATION

Abstract. In this article we constructed an asymptotic expansion of the solution undivided boundary value problem for singularly perturbed integro-differential equations with an initial jump phenomenon m – th order. We obtain the theorem about estimation of the remainder term's asymptotic with any degree of accuracy in the smallparameter.

Key words: singular perturbation, the integro-differential equation, a small parameter, asymptotic expansion, the initial jump, the boundary layer.

Introduction

Singularly perturbed equations act as mathematical models in many applied problems related to diffusion, heat and mass transfer, chemical kinetics and combustion, heat propagation in thin bodies, semiconductor theory, gyroscope motion, quantum mechanics, biology and biophysics and many other branches of science and technology. In this paper we consider general undivided boundary-value problem for singularly perturbed linear integro-differential equations of n-th order, when the boundary conditions are not ordered with respect to the highest derivatives. At first the characteristic features of the problem under consideration are that the limiting unperturbed problem degenerates incompletely, i.e. the loss of boundary conditions imposed on the initial perturbed problem does not occur and secondly, the solution of the singularly perturbed problem as the small parameter tends to zero tends to the solution of the unperturbed equation with changed boundary conditions. The values of the initial jumps of the solution and of the integral terms are determined. A uniform asymptotic expansion of the solutions of the original singularly perturbed integro-differential boundary value problem with any degree of accuracy with respect to the small parameter is constructed. The solution of the above problems made it possible to extend the class of singularly perturbed integro-differential equations possessing the phenomena of initial jumps. The scientific novelty of the presented work is that the presence of integrals qualitatively changes the asymptotic representation of the solution of the corresponding integro-differential equations.

Note that other mathematical school of singularly perturbed equations in Kazakhstan and abroad investigate only boundary value problems, which does not have an initial jump. In our previous works in [1-10], we considered the initial and boundary value problems that are equivalent to the Cauchy problem with the initial jump for differential and integro-differential equations in the stable case.

Consider the following singularly perturbed integro-differential equation

$$L_\varepsilon y \equiv \varepsilon y^{(n)} + A_1(t)y^{(n-1)} + ... + A_n(t)y = F(t) + \int_0^1 \sum_{i=0}^{m+1} H_i(t,x)y^{(i)}(x,\varepsilon)dx \qquad (1)$$

with nonlocal boundary conditions

$$h_i y(t,\varepsilon) \equiv \sum_{j=0}^{m} \alpha_{ij} y^{(j)}(0,\varepsilon) + \sum_{j=0}^{l} \beta_{ij} y^{(j)}(1,\varepsilon) = a_i, \quad i = \overline{1,n}, \; m < n-1, \; l < n-1, \qquad (2)$$

where $\varepsilon > 0$ is a small parameter, α_{ij}, β_{ij}, $a_i \in R$ are known constants independent of ε and $\alpha_{im} \neq 0, i = \overline{1,n}$.

Assume that the following conditions hold:

(C1) Functions $A_i(t), F(t), i = \overline{1,n}$ are sufficiently smooth and defined on the interval $[0,1]$.

(C2) $A_1(t) \geq \gamma = const > 0, \; 0 \leq t \leq 1$.

(C3) Functions $H_i(t,x), i = \overline{0,m+1}$ are defined in the domain $D = \{0 \leq t \leq 1, 0 \leq x \leq 1\}$ and sufficiently smooth.

(C4) $$\overline{\Delta} = \begin{vmatrix} h_1 y_{10}(t) & ... & h_1 y_{n-1,0}(t) & \alpha_{1m} \\ ... & ... & ... & ... \\ h_n y_{10}(t) & ... & h_n y_{n-1,0}(t) & \alpha_{nm} \end{vmatrix} \neq 0,$$

where $y_{i0}(t), \quad i = \overline{1,n-1}$ are the fundamental set of solutions of the following homogeneous differential equation

$$L_0 y(t) \equiv A_1(t)y^{(n-1)}(t) + ... + A_n(t)y\ (t) = 0.$$

(C5) $\lambda = 1$ is not an eigenvalue of the kernel $H(t,s,\varepsilon)$.

(C6) $$\overline{\omega} = \begin{vmatrix} 1+\overline{d}_{11} & \overline{d}_{12} & ... & \overline{d}_{1n} \\ \overline{d}_{21} & 1+\overline{d}_{22} & ... & \overline{d}_{2n} \\ ... & ... & ... & ... \\ \overline{d}_{n1} & \overline{d}_{n2} & ... & 1+\overline{d}_{nn} \end{vmatrix} \neq 0.$$

(C7) Number 1 is not an eigenvalue of the kernel $\overline{H(t,s)}$.

For the solution of the problem (1),(2) are valid the following limiting equalities:

$$\lim_{\varepsilon \to 0} y^{(j)}(t,\varepsilon) = \overline{y}^{(j)}(t), \; j = \overline{0,m-1}, \; 0 \leq t \leq 1, \qquad (3)$$

$$\lim_{\varepsilon\to 0} y^{(m+j)}(t,\varepsilon)=\overline{y}^{(m+j)}(t),\ j=\overline{0,n-1-m},\ 0<t\le 1,$$

where $\overline{y}(t)$ is the solution of the degenerate problem, Δ_0 is the initial jump of the solution,

$$L_0\overline{y}\equiv A_1(t)\overline{y}^{(n-1)}(t)+\sum_{i=2}^{n}A_i(t)\overline{y}^{(n-i)}(t)=F(t)+\int_0^1\sum_{i=0}^{m+1}H_i(t,x)\overline{y}^{(i)}(x)dx+\Delta_0 H_{m+1}(t,0),$$

$$h_i\overline{y}(t)\equiv\sum_{j=0}^{m}\alpha_{ij}\overline{y}^{(j)}(0)+\sum_{j=0}^{l}\beta_{ij}\overline{y}^{(j)}(1)=a_i-\alpha_{im}\Delta_0,\quad i=\overline{1,n}. \tag{4}$$

From (3) it follows that the solution $y(t,\varepsilon)$ of the general boundary value problem (1) and (2) converges to the solution $\overline{y}(t)$ of the modified degenerate problem (4) as $\varepsilon\to 0$. We note that the limits for $y^{(m+j)}(t,\varepsilon), j=\overline{0,n-1-m}$ are not uniform on the interval $0\le t\le 1$. They are uniform on the interval $0<t_0\le t\le 1$, where t_0 is sufficiently small but fixed number as $\varepsilon\to 0$. In the work will be constructed uniformly asymptotic expansion of the solution of the problem (1),(2) on the interval $0\le t\le 1$.

Since the solution of the problem (1) and (2) has the $m-$th order initial jump at the point $t=0$, we seek the asymptotic expansion of the solution of the problem (1), (2) in the next form:

$$y(t,\varepsilon)=y_\varepsilon(t)+\varepsilon^m w_\varepsilon(\tau),\quad \tau=\frac{t}{\varepsilon}, \tag{5}$$

where $y_\varepsilon(t)$ is a regular part of the asymptotic and $w_\varepsilon(\tau)$ is a boundary layer part, those can be represented in the form:

$$y_\varepsilon(t)=\sum_{i=0}^{\infty}\varepsilon^i y_i(t),\quad w_\varepsilon(\tau)=\sum_{i=0}^{\infty}\varepsilon^i w_i(\tau). \tag{6}$$

Substituting the series (5) into (1), we obtain the following equalities:

$$\varepsilon\left(y_\varepsilon^{(n)}(t)+\varepsilon^{m-n}\overset{(n)}{w}_\varepsilon(\tau)\right)+\sum_{i=1}^{n}A_i(t)\left(y_\varepsilon^{(i)}(t)+\varepsilon^{m-n+i}\overset{(i)}{w}_\varepsilon(\tau)\right)=$$
$$=F(t)+\int_0^1\sum_{i=0}^{m+1}H_i(t,x)\left(y_\varepsilon^{(i)}(x)+\varepsilon^{m-i}\overset{(i)}{w}_\varepsilon\left(\frac{x}{\varepsilon}\right)\right)dx. \tag{7}$$

By replacing the integral expression $s=\dfrac{x}{\varepsilon}$ on the right-hand side of the equation (7), we get the improper integral

$$J(t,\varepsilon)=\int_0^{\frac{1}{\varepsilon}}\sum_{i=0}^{m+1}\varepsilon^{m+1-i}H_i(t,\varepsilon s)\overset{(i)}{w}_\varepsilon(s)ds=\int_0^{\infty}\sum_{i=0}^{m+1}\varepsilon^{m+1-i}H_i(t,\varepsilon s)\overset{(i)}{w}_\varepsilon(s)ds-$$
$$-\int_{\frac{1}{\varepsilon}}^{\infty}\sum_{i=0}^{m+1}\varepsilon^{m+1-i}H_i(t,\varepsilon s)\overset{(i)}{w}_\varepsilon(s)ds. \tag{8}$$

The improper integral in (8) converges and the second sum in (8) is vanished, because $O\left(\exp\left(-\gamma\frac{t}{\varepsilon}\right)\right)$ is less than any power of ε, as $\varepsilon \to 0$.

We write separately the coefficients depending on t and on τ we obtain the following equalities for $y_\varepsilon(t)$ and $w_\varepsilon(\tau)$:

$$\varepsilon y_\varepsilon^{(n)}(t)+\sum_{i=1}^{n}A_i(t)y_\varepsilon^{(n-i)}(t)=F(t)+\int_0^1\sum_{i=0}^{m+1}H_i(t,x)y_\varepsilon^{(i)}(x)dx+\int_0^\infty\sum_{i=0}^{m+1}\varepsilon^{m+1-i}H_i(t,\varepsilon s)\overset{(i)}{w}_\varepsilon(s)ds, \tag{9}$$

$$\overset{(n)}{w}_\varepsilon(\tau)+A_1(\varepsilon\tau)\overset{(n-1)}{w}_\varepsilon(\tau)+\varepsilon A_2(\varepsilon\tau)\overset{(n-2)}{w}_\varepsilon(\tau)+\ldots+\varepsilon^{n-1}A_n(\varepsilon\tau)w_\varepsilon(\tau)=0. \tag{10}$$

By the degree of ε formally expanding $H_i(t,\varepsilon s), i=\overline{0,m+1}$ into a Taylor series at the point $(t,0)$:

$$H_i(t,\varepsilon s)=H_i(t,0)+\varepsilon sH_i'(t,0)+\frac{(\varepsilon s)^2}{2!}H_i''(t,0)+\ldots+\frac{(\varepsilon s)^k}{k!}H_i^{(k)}(t,0)+\ldots\quad i=\overline{0,m+1} \tag{11}$$

Use (11) in (9), equating coefficients of like powers of ε , for the regular part $y_k(t), k=0,1,2,\ldots.$ we arrive the following equalities:

$$A_1y_0^{(n-1)}(t)+\sum_{k=2}^{n}A_k(t)y_0^{(n-k)}(t)=F(t)+\int_0^1\sum_{i=0}^{m+1}H_i(t,x)y_0^{(i)}(x)dx+\int_0^\infty H_{m+1}(t,0)\overset{(m+1)}{w_0}(s)ds$$

where $\int_0^\infty H_{m+1}(t,0)\overset{(m+1)}{w_0}(s)ds=-H_{m+1}(t,0)\overset{(m)}{w_0}(0)$, denote by

$$\Delta_0(t)=H_{m+1}(t,0)\Delta_0,\ \Delta_0=-\overset{(m)}{w_0}(0). \tag{12_0}$$

for determining the coefficient $y_0(t)$, we obtain the integro-differential equation

$$A_1y_0^{(n-1)}(t)+\sum_{k=2}^{n}A_k(t)y_0^{(n-k)}(t)=F(t)+\int_0^1\sum_{i=0}^{m+1}H_i(t,x)y_0^{(i)}(x)dx+\Delta_0(t), \tag{13_0}$$

where $\Delta_0(t)$ is defined by formula (12_0).

For determining the coefficients $y_k(t), k=1,2,\ldots.$ we obtain the integro-differential equation

$$A_1(t)y_k^{(n-1)}(t)+\sum_{i=2}^{n}A_i(t)y_k^{(n-i)}(t)=F_k(t)+\int_0^1\sum_{i=0}^{m+1}H_i(t,x)y_k^{(i)}(x)dx+\Delta_k(t), \tag{13_k}$$

where

$$\Delta_k(t)=H_{m+1}(t,0)\Delta_k,\ \Delta_k=-\overset{(m)}{w_k}(0) \tag{12_k}$$

and $F_k(t)$ is known function, can be written as

$$F_k(t)=\int_0^\infty\sum_{j=1}^{k}\frac{s^j}{j!}H_{m+1}^{(j)}(t,0)\overset{(m+1)}{w}_{k-j}(s)ds+\int_0^\infty\sum_{i=1}^{k}\sum_{j=0}^{k-i}\frac{s^j}{j!}H_{m+1-i}^{(j)}(t,0)\overset{(m+1-i)}{w}_{k-i-j}(s)ds-y_{k-1}^{(n)}(t),$$

$$k=\overline{1,m+1} \tag{14}$$

$$F_k(t)=\int_0^\infty\sum_{j=1}^{k}\frac{s^j}{j!}H_{m+1}^{(j)}(t,0)\overset{(m+1)}{w}_{k-j}(s)ds+\int_0^\infty\sum_{i=1}^{m+1}\sum_{j=0}^{k-i}\frac{s^j}{j!}H_{m+1-i}^{(j)}(t,0)\overset{(m+1-i)}{w}_{k-i-j}(s)ds-y_{k-1}^{(n)}(t),$$

$$k>m+1.$$

The values $\Delta_k(t),\Delta_k, k\geq 0$ are called respectively *theinitial jumps of the integral terms and solutions.*

By the degree of ε formally expanding $A_i(\varepsilon\tau),\ i=\overline{1,n}$ into a Taylor series at the point 0:

$$A_i(\varepsilon\tau)=A_i(0)+\varepsilon\tau A_i'(0)+\frac{(\varepsilon\tau)^2}{2!}A_i''(0)+\ldots+\frac{(\varepsilon\tau)^k}{k!}A_i^{(k)}(0)+\ldots\quad i=\overline{1,n}. \tag{15}$$

Use (15) in (10), equating coefficients of like power of ε on both sides (10), wev get the equations for the boundary layer functions $w_k(\tau),\quad k=0,1,2,\ldots$

$$\overset{(n)}{w}_0(\tau)+A_1(0)\overset{(n-1)}{w}_0(\tau)=0, \tag{16$_0$}$$

$$\overset{(n)}{w}_k(\tau)+A_1(0)\overset{(n-1)}{w}_k(\tau)=\Phi_k(\tau), \tag{16$_k$}$$

where $\Phi_k(\tau)$ is known function, can be written as

$$\Phi_k(\tau)=\begin{cases}-\sum_{j=1}^{k}\frac{\tau^j}{j!}A_1^{(j)}(0)\overset{(n-1)}{w}_{k-j}(\tau)-\sum_{m=0}^{k-1}\sum_{j=0}^{m}\frac{\tau^j}{j!}A_{k+1-m}^{(j)}(0)\overset{(n-1+m-k)}{w}_{m-j}(\tau),\ k=\overline{1,n-1},\\[2ex] -\sum_{j=1}^{k}\frac{\tau^j}{j!}A_1^{(j)}(0)\overset{(n-1)}{w}_{k-j}(\tau)-\sum_{m=k+1-n}^{k-1}\sum_{j=0}^{m}\frac{\tau^j}{j!}A_{k+1-m}^{(j)}(0)\overset{(n-1+m-k)}{w}_{m-j}(\tau),\quad k\geq n\end{cases} \tag{17}$$

To determineuniquely the terms $y_k(t)$ and $w_k(\tau)$ of the asymptotic, we use (5) in (6) and taking into account boundary condition (2)

$$\sum_{j=0}^{m}\alpha_{ij}[y_0^{(j)}(0)+\varepsilon y_1^{(j)}(0)+\ldots+\varepsilon^{m-j}(\overset{(j)}{w}_0(0)+\varepsilon\overset{(j)}{w}_1(0)+\ldots)]+$$

$$+\sum_{j=0}^{l}\beta_{ij}\left[y_0^{(j)}(1)+\varepsilon y_1^{(j)}(1)+\ldots+\varepsilon^{m-j}\left(\overset{(j)}{w}_0\left(\frac{1}{\varepsilon}\right)+\varepsilon\overset{(j)}{w}_1\left(\frac{1}{\varepsilon}\right)+\ldots\right)\right]=a_i\ ,\ i=\overline{1,n}. \tag{18}$$

In (18) $\overset{(j)}{w}_k\left(\frac{1}{\varepsilon}\right), k=0,1,\ldots$ it is not take into account, it can not be compared than any degree of ε.

Equating the coefficients at zero degrees of ε in (18) and in view of (12_0), we have

$$h_i y_0(t) = a_i + \alpha_{im}\Delta_0, \qquad i=\overline{1,n}. \tag{19_0}$$

Thus, the main coefficient $y_0(t)$ of the regular part of the asymptotic and the initial jump of the solution Δ_0 are determined from the problem (13_0), (19_0).

For determining the coefficient $w_0(\tau)$, we have the initial condition $\Delta_0 = -\overset{(m)}{w}_0(0)$ from (13_0), (19_0). Finding the missed initial condition for coefficient $w_0(\tau)$ we reduce the order of theequation (16_0) byintergrating from τ to ∞ and by virtue of the conditions $\overset{(i)}{w}_0(\infty)=0, i=\overline{0,n-1}$. As a result, after $n-1-m$ -th step, we obtain equation $\overset{(m+1)}{w}_0(\tau) + A_1(0)\overset{(m)}{w}_0(\tau)=0$. From this equation as $\tau=0$, we determine the initial condition $\overset{(m+1)}{w}_0(0) = -A_1(0)\overset{(m)}{w}_0(0)$.Continuing this process lowering the degree of equation (16_0), we obtain the following initial conditions for $w_0(\tau)$:

$$\overset{(i)}{w}_0(0) = (-1)^{m+1-i}\frac{\Delta_0}{A_1(0))^{m-i}}, \quad i=\overline{0,n-1}. \tag{20_0}$$

Thus, the main coefficient $w_0(\tau)$ of the boundary layer part of the asymptotic is determined from the problem (16_0), (20_0).

Thus, the zeroth approximation of the asymptotic expansion is completely constructed.

In the k -th approximation, for determining the boundary conditions of the coefficient $y_k(t)$, $k=1,2,\ldots$, we compare the coefficients of the same powersof the parameter ε. As a result, we obtain the following initial conditions for $y_k(t)$:

$$h_i y_k(t) = \begin{cases} \alpha_{im}\Delta_k - \sum\limits_{j=1}^{k}\alpha_{i,m-j}\overset{(m-j)}{w}_{k-j}(0), & k=\overline{1,m}, \\ \alpha_{im}\Delta_k - \sum\limits_{j=1}^{m}\alpha_{i,m-j}\overset{(m-j)}{w}_{k-j}(0), & k\geq m+1 \end{cases} \qquad i=\overline{1,n}. \tag{19_k}$$

From (13_k), (19_k) we determine $y_k(t), \Delta_k, k\geq 1$.

Now, we will be determine the initial conditions for the coefficient $w_k(\tau), k\geq 1$. In order to find themissingof the equation (16_k) by virtue of the conditions $\overset{(i)}{w}_k(\infty)=0, i=\overline{0,n-1}$. Then, we get the initial conditions for determining $w_k(\tau), k\geq 1$:

$$\overset{(i)}{w}_k(0)=\begin{cases}\dfrac{(-1)^{m-i+1}}{A_1^{m-i}(0))}\Delta_k+(-1)^{n-1-i}\int\limits_0^\infty\sum\limits_{j=n-1-m}^{n-2-i}\dfrac{s^j}{j!}(A_1(0))^{j-(n-1-i)}\Phi_k(s)ds, \quad i=\overline{0,m},\\ \\ (-1)^{i-m+1}A_1^{i-m}(0)\Delta_k+(-1)^{n-i}\int\limits_0^\infty\sum\limits_{j=n-1-i}^{n-2-m}\dfrac{s^j}{j!}(A_1(0))^{j-(n-1-i)}\Phi_k(s)ds, i\ge m+1\end{cases} \tag{20_k}$$

Thus, the k -th approximation of the asymptotic is completely constructed.

Theorem. *Let functions* $A_i(t), F(t)\in C^{N+n-m}[a,b], i=\overline{1,n}$ *and conditions(C2) - (C7) hold. Then for sufficiently small* ε *the boundary value problem (1) and (2) has an unique solution on the* $0\le t\le 1$ *and that is expressed by the formula*

$$y(t,\varepsilon)=\bar{y}_N(t,\varepsilon)+R_N(t,\varepsilon), \tag{21}$$

where $\bar{y}_N(t,\varepsilon)$ *is defined by the formula*

$$\bar{y}_N(t,\varepsilon)=\sum_{k=0}^{N}\varepsilon^k y_k(t)+\varepsilon^m\sum_{k=0}^{N+n-1-m}\varepsilon^k w_k(\tau), \quad \tau=\frac{t}{\varepsilon}, \tag{22}$$

and for the remainder term the estimates are valid

$$\left|R_N^{(i)}(t,\varepsilon)\right|\le C\varepsilon^{N+1}, \quad i=\overline{0,n-1}, \qquad 0\le t\le 1. \tag{23}$$

where $C>0$ *is a some constant independent of* ε .

Proof. We construct the N -th partial sum (22) of the expansion (5),(6).

The function $\bar{y}_N(t,\varepsilon)$ satisfies problem (1), (2) with accuracy of order $O(\varepsilon^{N+1})$, i.e.

$$L_\varepsilon\bar{y}_N(t,\varepsilon)=F(t)+\int_0^1\sum_{i=0}^{m+1}H_i(t,x)\bar{y}_N^{(i)}(x,\varepsilon)dx+O(\varepsilon^{N+1}), \tag{24}$$

$$h_i\bar{y}_N(t,\varepsilon)=a_i+O(\varepsilon^{N+1}), \quad i=1,n$$

Denote by $y(t,\varepsilon)=\bar{y}_N(t,\varepsilon)+R_N(t,\varepsilon)$. Then for the remainder $R_N(t,\varepsilon)$ we obtain the problem as follows

$$L_\varepsilon R_N(t,\varepsilon)=\int_0^1\sum_{i=0}^{m+1}H_i(t,x)R_N^{(i)}(x,\varepsilon)dx+O(\varepsilon^{N+1}), \tag{25}$$

$$h_iR_N(t,\varepsilon)=O(\varepsilon^{N+1}), \quad i=1,n \ .$$

We apply the asymptotic estimation of the solution of the problem (1),(2) to the problem (25). Then we obtain the estimates

$$\left|R_N^{(j)}(t,\varepsilon)\right| \le C\varepsilon^{N+1} + C\varepsilon^{N+1+m-j}\exp\left(-\gamma\frac{t}{\varepsilon}\right), \quad j=\overline{0,n-1}. \tag{26}$$

This means that estimates $R_N^{(m+1)}(t,\varepsilon)=O(\varepsilon^N),\dots,R_N^{(n-1)}(t,\varepsilon)=O(\varepsilon^{N-n+2+m})$is valid at point $t=0$, i.e. The required estimates do not hold. To obtain the necessary estimates, we consider the equalities

$$y^{(m+1)}(t,\varepsilon)=y_N^{(m+1)}(t,\varepsilon)+R_N^{(m+1)}(t,\varepsilon),\ y^{(m+1)}(t,\varepsilon)=y_{N+1}^{(m+1)}(t,\varepsilon)+R_{N+1}^{(m+1)}(t,\varepsilon) \tag{27}$$

Hence, equating the right-hand sides of (27), we get

$$R_N^{(m+1)}(t,\varepsilon)=y_{N+1}^{(m+1)}(t,\varepsilon)-y_N^{(m+1)}(t,\varepsilon)+R_{N+1}^{(m+1)}(t,\varepsilon), \tag{28}$$

where $\overline{y}_{N+1}^{(m+1)}(t,\varepsilon)-\overline{y}_N^{(m+1)}(t,\varepsilon)=\varepsilon^{N+1}y_{N+1}^{(m+1)}(t)+\varepsilon^{N+n-1-m}\overset{(m+1)}{w}_{N+n-m}(\tau)$and the remainder term $R_{N+1}^{(m+1)}(t,\varepsilon)$ in (28) satisfies the estimate$\left|R_{N+1}^{(m+1)}(t,\varepsilon)\right|\le C\varepsilon^{N+2}+C\varepsilon^{N+1}\exp\left(-\gamma\frac{t}{\varepsilon}\right)$. Thus, we obtain the required estimates: $\left|R_N^{(m+1)}(t,\varepsilon)\right|\le C\varepsilon^{N+1}$. Similarly, considering the equalities

$$y^{(n-1)}(t,\varepsilon)=y_N^{(n-1)}(t,\varepsilon)+R_N^{(n-1)}(t,\varepsilon),\ y^{(n-1)}(t,\varepsilon)=y_{N+n-1-m}^{(n-1)}(t,\varepsilon)+R_{N+n-1-m}^{(n-1)}(t,\varepsilon) \tag{29}$$

Hence, equating the right-hand sides of (29), we obtain

$$R_N^{(n-1)}(t,\varepsilon)=y_{N+n-1-m}^{(n-1)}(t,\varepsilon)-y_N^{(n-1)}(t,\varepsilon)+R_{N+n-1-m}^{(n-1)}(t,\varepsilon), \tag{30}$$

where

$$\overline{y}_{N+n-1-m}^{(n-1)}(t,\varepsilon)-\overline{y}_N^{(n-1)}(t,\varepsilon)=\varepsilon^{N+1}y_{N+1}^{(n-1)}(t)+\ldots+\varepsilon^{N+n-1-m}\overset{(n-1)}{y}_{N+n-1-m}(t)+$$

$$+\varepsilon^{N+1}\overset{(n-1)}{w}_{N+n-m}(\tau)+\ldots+\varepsilon^{N+n-1-m}\overset{(n-1)}{w}_{N+2(n-1-m)}(\tau),$$

The remainder term $R_{N+n-1-m}^{(n-1)}(t,\varepsilon)$ in (30) satisfies the estimate $\left|R_{N+n-1-m}^{(n-1)}(t,\varepsilon)\right|\le C\varepsilon^{N+n-m}+C\varepsilon^{N+1}\exp\left(-\gamma\frac{t}{\varepsilon}\right)$. Thus, we obtain the required estimates $\left|R_N^{(n-1)}(t,\varepsilon)\right|\le C\varepsilon^{N+1}$. Theorem is proved.

CONCLUSION

We investigatedasymptotic expansion of solution of general boundary value problem with initial jumps for higher-order singularly perturbed integro-differential equation with any degree of accuracy with respect to a small parameter have been constructed.

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ЖОҒАРҒЫ РЕТТІ СИНГУЛЯРЛЫ АУЫТҚЫҒАН ИНТЕГРАЛДЫ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУ ҮШІН ЖАЛПЫЛАНҒАН БАСТАПҚЫ СЕКІРІСТІ ШЕТТІК ЕСЕБІ ШЕШІМІНІҢ АСИМПТОТИКАЛЫҚ ЖІКТЕЛУІ

Аннотация. Мақаладасингулярлы ауытқығанинтегралды-дифференциалдық теңдеулер үшін*m*-ші ретті бастапқы секірісі бар бөлінбеген шеттік есепшешімінің асимптотикалық жіктелуі құрылды. Кіші параметр бойынша кезкелген дәлдікпен асимптотиканың қалдық мүшесін бағалау туралы теорема алынды.

Түйін сөздер: сингулярлы ауытқу, интегралды-дифференциалдық теңдеу, кіші параметр, асимптотика-лық жіктелу, бастапқысекіріс, шекаралық қабат.

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АСИМПТОТИЧЕСКОЕ РАЗЛОЖЕНИЕ РЕШЕНИЯ ОБЩЕЙ КРАЕВОЙ ЗАДАЧИ С НАЧАЛЬНЫМИ СКАЧКАМИ ДЛЯ ВЫСШЕГО ПОРЯДКА СИНГУЛЯРНО ВОЗМУЩЕННОЕ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОЕ УРАВНЕНИЕ

Аннотация. В статье построено асимптотическое разложение решений неразделенной краевой задачи с начальным скачком *m*-го порядка для сингулярно возмущенных интегро-дифференциальных уравнений. Полученатеорема об оценке остаточного члена асимптотики с любой степенью точности по малому параметру.

Ключевые слова: сингулярное возмущение, интегро-дифференциальное уравнение, малый параметр, асимптотическое разложение, начальный скачок, погранслой.

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