NEWS
OF THENATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
PHYSICO-MATHEMATICAL SERIES
ISSN 1991-346X **https://doi.org/10.32014/2020.2518-1726.78**
Volume 5, Number 333 (2020), 19 – 27

UDC 539.12-17; 52:531.51; 52:530.12
IRSTI 29.05.41; 41.17.41

K. Boshkayev[1,2], A. Malybayev[1], H. Quevedo[1,3,4],
G. Nurbakyt[1], A. Taukenova[1,2], A. Urazalina[1,2]

[1]Department of Theoretical and Nuclear Physics, Al-Farabi Kazakh National University, Almaty, Kazakhstan;
[2]National Nanotechnology Laboratory of Open Type, Almaty, Kazakhstan;
[3]Dipartimento di Fisica and ICRA, Universita di Roma "La Sapienza", Roma, Italy;
[4]Instituto de Ciencias Nucleares, Universidad Nacional Autonoma de Mexico, Mexico.
E-mail: kuantay@mail.ru; algis_malybayev@mail.ru; quevedo@nucleares.unam.mx;
gumi-nur@mail.ru; aliya_tauken@mail.ru; y.a.a.707@mail.ru

THE CORRESPONDENCE OF THE EREZ-ROSEN SOLUTION WITH THE HARTLE-THORNE SOLUTION IN THE LIMITING CASE OF $\sim Q$ AND $\sim M^2$

Abstract. The link between exterior solutions to the Einstein gravitational field equations such as the exact Erez-Rosen metric and approximate Hartle-Thorne metric is established here for the static case in the limit of linear mass quadrupole moment (Q) and second order terms in total mass (M). To this end, the Geroch-Hansen multipole moments are calculated for the Erez-Rosen and Hartle-Thorne solutions in order to find the relationship among the parameters of both metrics. The coordinate transformations are sought in a general form with two unknown functions in the corresponding limit of $\sim Q$ and $\sim M^2$. By employing the perturbation theory, the approximate Erez-Rosen metric is written in the same coordinates as the Hartle-Thorne metric. By equating the radial and azimuthal components of the metric tensor of both solutions the sought functions are found in a straightforward way. It is shown that the approximation $\sim Q$ and $\sim M^2$, which is used throughout the article, is physical and suitable for solving most problems of celestial mechanics in post-Newtonian physics. This approximation does not require the use of the Zipoy-Voorhees transformation, which is a necessary strict mathematical requirement in the $\sim Q$ approximation, i.e. when no other approximations are made. This implies that the explicit form of the coordinate transformations depends entirely on the approximation that is adopted in each particular case. The results obtained here are in agreement with the previous results in the literature and can be applied to different astrophysical goals. The paper pursues not only pure scientific, but also academic purposes and can be used as an auxiliary and additional material to the special courses of general theory of relativity, celestial mechanics and relativistic astrophysics.

Keywords: exact and approximate solutions of Einstein's gravitational field equations, Erez-Rosen metric, Hartle-Thorne metric, coordinate transformations, quadrupole moment, Geroch-Hansen multipole moments, perturbation method.

1. Introduction.

There are a plenty of exact and approximate solutions to the Einstein field equations (EFE) in the literature [1,2]. Most of the solutions are pure mathematical and only some of them are physical and usually used in a various realistic astrophysical context. We here focus on the exterior exact Erez-Rosen (ER) [3] and approximate Hartle-Thorne (HT) [4,5] solutions, hich are very well established and widely exploited. The Erez-Rosen solution mainly involved in the description of the exterior gravitational field of a deformed astrophysical object. Instead the Hartle-Thorne solution is used to study both interior and exterior fields of slowly rotating and slightly deformed astrophysical objects in the strong field regime. In connection with this, it is interesting to show how to find the relationship between these solutions in the limiting static case with a small deformation.

The metric of a nonrotating mass with a quadrupole moment has been obtained by Erez and Rosen in 1959 [3] with a use of Weyl method [6]. This metric was also analyzed by applying the spheroidal coordinates, which are adapted to characterize the gravitational field of non-spherically symmetric bodies. Later corrected for several numerical coefficients by Doroshkevich (1966) [7], Winicour et al. (1968) [8] and Young and Coulter (1969) [9]. The physical properties of the ER metric were investigated by Zeldovich and Novikov [10] and later by Quevedo and Parkes [11]. More general solutions involving multipole moments were obtained by Quevedo, Quevedo and Mashhoon (QM) [12–15].

In the article dated back 1967 Hartle put forward his approach for studying physical properties of slowly rotating relativistic stars [4]. Physical quantities that describe the equilibrium configurations of rotating stars such as the change in mass, gravitational potential, eccentricity, binding energy, change in moment of inertia, quadrupole moment, etc. were proportional to the square of the star's angular velocity Ω^2. Hartle and Thorne tested the formalism for different equations of state of relativistic objects [5]. From that moment this solution is widely known as the Hartle-Thorne (HT) solution. Unlike other solutions of the Einstein equations, the Hartle-Thorne solution has an internal counterpart [4,16], which makes it more practical for investigation the equilibrium structure and physical characteristics of relativistic compact objects such as white dwarfs, neutron stars and hypothetical quark stars [17–21]. Relatively recently this solution was extended up to Ω^4 approximation [22].

The main objective of this article is to find the relationship between Hartle-Thorne solution and Erez-Rosen solution and show their equivalence in the limiting static case with a small deformation. We adopted the signature of the line elements for this article as $(+---)$ and used geometrical units $G = c = 1$.

It should be emphasized that the relationship between the ER and HT solutions has been established by Mashhoon and Theiss in 1991 [23], involving the Zipoy-Voorhees transformation in the limiting static case for small deformation. In addition Frutos-Alfaro and Soffel has shown that in the limit of $\sim Q$ and $\sim M^2$ for static case one can find the relationship between the two metrics without involving the Zipoy-Voorhees transformation [24]. In [25] we revisited the derivation by Mashhoon and Theiss providing all technical details in an instructive way. However, in this work we revisit the results of Frutos-Alfaro and Soffel [24], justifying physical significance, providing technical details. The paper pursues pure scientific and academic purposes.

The work is organized as follows. We review the main properties of the ER solution in section 2. The main physical characteristics of the exterior Hartle-Thorne solution are discussed in section 3. Section 4 is devoted to the computation of the multipole structure of the solutions. The linearized, up to the first order in mass quadrupole moment Q and to the second order in total mass M the Erez-Rosen solution is considered in section 5. The Hartle-Thorne solution in the limit of $\sim Q$ and $\sim M^2$ is considered in section 6. Using the perturbation method, the coordinate transformations are sought in section 7. Finally, we summarize our conclusions and discuss about future prospects.

2. The Erez-Rosen metric

The Erez-Rosen metric is an exact exterior solution with mass (m) and quadrupole (q) parameters that describes the gravitational field of static deformed objects in the strong field regime [26]. It belongs to the Weyl class of static axisymmetric vacuum solutions in prolate spheroidal coordinates (t, x, y, φ), with $x \geq 1$ and $-1 \leq y \leq 1$:

$$ds^2 = e^{2\psi}dt^2 - m^2 e^{-2\psi}\left[e^{2\gamma}(x^2 - y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (x^2 - 1)(1 - y^2)d\varphi^2\right], \quad (1)$$

where the metric functions ψ and γ depend on the spatial coordinates x and y, only, and m represents the mass parameter.

The solution found by Erez and Rosen has the following form [27]

$$\psi = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + \frac{1}{2}q(3y^2 - 1)\left[\frac{1}{4}(3x^2 - 1)\ln\left(\frac{x-1}{x+1}\right) + \frac{3}{2}x\right] \quad (2)$$

and

$$\gamma = \frac{1}{2}(1 + q)^2\ln\left(\frac{x^2-1}{x^2-y^2}\right) - \frac{3}{2}q(1 - y^2)\left[x\ln\left(\frac{x-1}{x+1}\right) + 2\right] + \quad (3)$$

$$+\frac{9}{16}q^2(1-y^2)\Bigg[x^2+4y^2-9x^2y^2-\frac{4}{3}$$
$$+x\left(x^2+7y^2-9x^2y^2-\frac{5}{3}\right)\ln\left(\frac{x-1}{x+1}\right)$$
$$+\frac{1}{4}(x^2-1)(x^2+y^2-9x^2y^2-1)\ln^2\left(\frac{x-1}{x+1}\right)\Bigg]$$

where q is the quadrupole parameter.

3. The exterior Hartle-Thorne solution

The general form of the exterior approximate HT metric [4, 5] in spherical $(\mathrm{t}, \mathrm{R}, \Theta, \phi)$ coordinates is given by:

$$ds^2=\left(1-\frac{2M}{R}\right)\left[1+2k_1P_2(\cos\Theta)-2\left(1-\frac{2M}{R}\right)^{-1}\frac{J^2}{R^4}(2\cos^2\Theta-1)\right]dt^2$$
$$-\left(1-\frac{2M}{R}\right)^{-1}\left[1-2\left(k_1-\frac{6J^2}{R^4}\right)P_2(\cos\Theta)-2\left(1-\frac{2M}{R}\right)^{-1}\frac{J^2}{R^4}\right]dR^2 \quad (4)$$
$$-R^2[1-2k_2P_2(\cos\Theta)](d\Theta^2+\sin^2\Theta\, d\phi^2)+\frac{4J}{R}\sin^2\Theta\, dt\, d\phi$$

where

$$k_1=\frac{J^2}{MR^3}\left(1+\frac{M}{R}\right)+\frac{5}{8}\frac{Q-J^2/M}{M^3}Q_2^2(x) \quad (5)$$

$$k_2=k_1+\frac{J^2}{R^4}+\frac{5}{4}\frac{Q-J^2/M}{M^2R}\left(1-\frac{2M}{R}\right)^{-\frac{1}{2}}Q_2^1(x) \quad (6)$$

and

$$Q_2^1(x)=(x^2-1)^{\frac{1}{2}}\left[\frac{3x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^2-2}{x^2-1}\right]$$

$$Q_2^2(x)=(x^2-1)\left[\frac{3x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^3-5x}{(x^2-1)^2}\right]$$

are the associated Legendre functions of the second kind, being $P_2(\cos\Theta)=(1/2)(3\cos^2\Theta-1)$ the Legendre polynomial, and $x=\frac{R}{M}-1$. The constants M, J and Q are the total mass, angular momentum and quadrupole moment of a rotating object, respectively. Note, that according to Hartle $Q>0$ for oblate and $Q<0$ for prolate objects.

In order to obtain the HT solution for static objects, we set $J=0$ in the general form (4) and obtain

$$ds^2=\left(1-\frac{2M}{R}\right)[1+2k_1P_2(\cos\Theta)]dt^2-\left(1-\frac{2M}{R}\right)^{-1}[1-2k_1P_2(\cos\Theta)]dR^2$$
$$-R^2[1-2k_2P_2(\cos\Theta)](d\Theta^2+\sin^2\Theta\, d\phi^2) \quad (7)$$

Where

$$k_1=\frac{5}{8}\frac{Q}{M^3}Q_2^2(x) \quad (8)$$

$$k_2=k_1+\frac{5}{4}\frac{Q}{M^2R}\left(1-\frac{2M}{R}\right)^{-\frac{1}{2}}Q_2^1(x) \quad (9)$$

Hence we can find the function $\psi(x,y)$ for the static HT metric from a simple relation

$$g_{tt}=e^{2\psi} \quad (10)$$

The solution found for ψ has the following form

$$\psi = \frac{1}{2}\ln\left[\left(\frac{x-1}{x+1}\right)\left(1 - \frac{5Q(3y^2-1)\left(10x-6x^3+3(x^2-1)^2\ln\left[\frac{x+1}{x-1}\right]\right)}{16M^3(1-x^2)}\right)\right] \tag{11}$$

where $y = \cos\Theta$. Finally, for the function γ it is not possible to find an explicit expression, unless we consider the approximate version of the ER metric. This will be done in Sec. 5.

4. Geroch-Hansen multipole moments

Using the original definition formulated by Geroch [28], the calculation of multipole moments is quite laborious. Fodor et al. [29] found a relation between the Ernst potential [30,31] and the multipole moments which facilitates the computation. In the case of static axisymmetric space–times, the Ernst potential is defined as

$$\xi(x,y) = \frac{1-e^{2\psi}}{1+e^{2\psi}} \tag{12}$$

The idea is that the multipole moments can be obtained explicitly from the values of the Ernst potential on the axis by using the following procedure. On the axis of symmetry $y = 1$, we can introduce the inverse of theWeyl coordinate z as

$$\tilde{z} = \frac{1}{z} = \frac{1}{mx} \tag{13}$$

If we introduce the inverse potential as

$$\tilde{\xi}(\tilde{z},1) = \frac{1}{z}\xi(\tilde{z},1) \tag{14}$$

The multipole moments can be calculated as

$$M_n = m_n + d_n, \qquad m_n = \frac{1}{n!}\frac{d^n\tilde{\xi}(\tilde{z},1)}{d\tilde{z}^n} \tag{15}$$

where $d_n$must be determined from the original Geroch definition (e.g. Refs. [15]). For the Erez–Rosen metric, the Geroch-Hansen multipole moments read

$$M_0 = m, \qquad M_2 = \frac{2}{15}qm^3 \tag{16}$$

where M_0 is the monopole moment and M_2 is the quadrupole moment.

For the Hartle–Thorne metric, we obtain

$$M_0 = M, \qquad M_2 = -Q \tag{17}$$

As one can see from the Geroch-Hansen definition of multipole moments the quadrupole moment of the HT metric has an opposite sign, which is due to the use of a different convention.

5. The approximated Erez-Rosen solution in the limit of $\sim Q$ and $\sim M^2$

In order to obtain the approximated ER metric we express m and q in terms of M and Q by equating relations (16) and (17), respectively as follows

$$m = M, \qquad q = -\frac{15}{2}\frac{Q}{M^3} \tag{18}$$

and find its limit in $\sim Q$ and $\sim M^2$ by expanding in Taylor series keeping only Q and M^2 and neglecting QM^2 terms. Taking into account $x = \frac{r}{m} - 1$ and $y = \cos\theta$, the final result is written in spherical-like coordinates (t, r, θ, ϕ)

$$ds^2 = \left(1-\frac{2M}{r}+\frac{2QP_2(\cos\theta)}{r^3}+\frac{2MQP_2(\cos\theta)}{r^4}\right)dt^2 \quad (19)$$
$$-\left(1+\frac{2M}{r}+\frac{4M^2}{r^2}-\frac{2QP_2(\cos\theta)}{r^3}-\frac{2MQ(5P_2^2(\cos\theta)+11P_2(\cos\theta)-1)}{3r^4}\right)dr^2$$
$$-r^2\left(1-\frac{2QP_2(\cos\theta)}{r^3}-\frac{2MQ(5P_2^2(\cos\theta)+5P_2(\cos\theta)-1)}{3r^4}\right)d\theta^2$$
$$-r^2\sin^2\theta\left(1-\frac{2QP_2(\cos\theta)}{r^3}-\frac{6MQP_2(\cos\theta)}{r^4}\right)d\phi^2$$

6. The Hartle-Thorne solution in the limit of $\sim Q$ and $\sim M^2$

In order to obtain the HT metric for static objects we set $J=0$ and find its limit in $\sim Q$ and $\sim M^2$, taking into account $x=\frac{R}{M}-1$ and $y=\cos\Theta$. So the HT metric in standard spherical coordinates (t,R,Θ,ϕ) reads

$$ds^2 = \left(1-\frac{2M}{R}+\frac{2QP_2(\cos\Theta)}{R^3}+\frac{2MQP_2(\cos\Theta)}{R^4}\right)dt^2-\left(1+\frac{2M}{R}+\frac{4M^2}{R^2}-\frac{2QP_2(\cos\Theta)}{R^3}-\frac{10MQP_2(\cos\Theta)}{R^4}\right)dR^2 \quad (20)$$
$$-R^2\left(1-\frac{2QP_2(\cos\Theta)}{R^3}-\frac{5MQP_2(\cos\Theta)}{R^4}\right)(d\Theta^2+\sin^2\Theta d\phi^2)$$

7. Coordinate transformations from the Erez-Rosen metric to the Hartle-Thorne metric

To obtain the correspondence between the ER solution, with coordinates (t,r,θ,ϕ), and the HT solution, with coordinates (t,R,Θ,ϕ), both solutions must be written in the same coordinates. Therefore, we search for a coordinate transformation of the following form:

$$r\to R+\frac{MQ}{R^3}f_1(\Theta) \qquad \theta\to\Theta+\frac{MQ}{R^4}f_2(\Theta) \quad (21)$$

where $f_1(\Theta)$and $f_2(\Theta)$ are the sought unknown functions. In view of $\sim Q$ and $\sim M^2$ approximation the functions f_1 and f_2 depend only on Θ. The total differentials of the coordinates are given by:

$$dr=\frac{\partial r}{\partial R}dR+\frac{\partial r}{\partial\Theta}d\Theta=\left(1-\frac{3MQ}{R^4}f_1(\Theta)\right)dR+\frac{MQ}{R^3}\left(\frac{\partial f_1(\Theta)}{\partial\Theta}\right)d\Theta \quad (22)$$
$$d\theta=\frac{\partial\theta}{\partial R}dR+\frac{\partial\theta}{\partial\Theta}d\Theta=\left(-\frac{4MQ}{R^5}f_2(\Theta)\right)dR+\left(1+\frac{MQ}{R^4}\frac{\partial f_2(\Theta)}{\partial\Theta}\right)d\Theta \quad (23)$$

These expressions should be plugged in the approximated ER solution (19). Then, only terms $\sim Q$ and $\sim M^2$ must be retained. We thus obtain the approximated ER metric in the same coordinates as the HT solution with the same parameters

$$ds^2=\left(1-\frac{2M}{R}+\frac{2QP_2(\cos\Theta)}{R^3}+\frac{2MQP_2(\cos\Theta)}{R^4}\right)dt^2$$
$$-\left(1+\frac{2M}{R}+\frac{4M^2}{R^2}-\frac{2QP_2(\cos\Theta)}{R^3}-\frac{2MQ(5P_2^2(\cos\Theta)+11P_2(\cos\Theta)-9f_1(\Theta)-1)}{3R^4}\right)dR^2$$
$$-R^2\left(1-\frac{2QP_2(\cos\Theta)}{R^3}-\frac{2MQ\left(5P_2(\cos\Theta)(1+P_2(\cos\Theta))+3\left(f_1(\Theta)-f_2'(\Theta)\right)-1\right)}{3R^4}\right)d\Theta^2 \quad (24)$$
$$+\left(\frac{2MQ\left(f_1'(\Theta)-4f_2(\Theta)\right)}{R^3}\right)dRd\Theta$$
$$-R^2\sin^2\Theta\left(1-\frac{2QP_2(\cos\Theta)}{R^3}-\frac{2MQ(3P_2(\cos\Theta)+f_1(\Theta)+f_2(\Theta)\cot\Theta)}{R^4}\right)d\phi^2$$

Furthermore, by equating the corresponding g_{RR} components of the metric tensor of both approximated ER (24) and HT (20) solutions, written in (t, R, Θ, ϕ) coordinates, we find the expression for function $f_1(\Theta)$ as follows

$$f_1(\Theta) = \frac{5P_2^2(\cos\Theta) - 4P_2(\cos\Theta) - 1}{9} \tag{25}$$

Analogously, by comparing only the azimuthal components of the metric tensor $g_{\phi\phi}$ of both solutions, we find the function $f_2(\Theta)$ as

$$f_2(\Theta) = \frac{1}{6}\big(2 - 5P_2(\cos\Theta)\big)\cos\Theta\sin\Theta \tag{26}$$

To this end, if we plug these functions into the mixed component of the metric tensor $g_{R\Theta}$ of the approximated ER solution (24), written in (R, Θ) coordinates, $g_{R\Theta}$ vanishes, as we expected.

8. Conclusion. We explored the Erez-Rosen and Hartle–Thorne metrics (in the absence of rotation) in the limit $\sim Q$ and $\sim M^2$ by using the perturbation method. The approximation that we used throughout the paper is physical and convenient to solve most problems of celestial mechanics in the post-Newtonian Physics. We showed that the approximate Erez–Rosen line element coincides with the Hartle–Thorne solution in the considered limit.

The use of Geroch and Hansen invariant definition of the multipole moments helped us to calculate the corresponding mass monopole and quadrupole moments and establish the interconnection among the parameters of both solutions.

In addition, we have showed that the explicit form of the coordinate transformations in the limit $\sim Q$ and $\sim M^2$ do not require the use of the Zipoy-Voorhees transformation in view of Ref. [32] (see also Ref. [25] for details).

Due to the recent results [33–36], it will be interesting to find the connection between the Erez–Rosen and Zipoy-Voorhees (q-metric) solutions. This will be the issue of future studies.

Acknowledgments: This work was supported by the Ministry of Education and Science of the Republic of Kazakhstan, Program IRN: BR05236494 and Grant IRN: AP08052311.

К. Бошкаев[1,2], А. Малыбаев[1], Э. Кеведо[1,3,4],
Г. Нұрбақыт[1], А. Таукенова[1,2], А. Уразалина[1,2]

[1]Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан;
[2]Ашық түрдегі ұлттық нанотехнологиялық зертхана, Алматы, Қазақстан;
[3]Ла Сапиенца Рим университеті, Рим, Италия;
[4]Мексика ұлттық автономиялық университеті, Ядролық ғылымдар институты, Мехико, Мексика

$\sim Q$ ЖӘНЕ $\sim M^2$ ЖУЫҚТАУЫНДА ЭРЕЗ-РОЗЕН ЖӘНЕ ХАРТЛ-ТОРН МЕТРИКАЛАРЫНЫҢ СӘЙКЕСТІГІ

Аннотация. Бұл жұмыста гравитациялық өріс үшін Эйнштейн теңдеулерінің сыртқы шешімі, атап айтқанда Эрез-Розен және Хартл-Торн метрикалары арасына байланыс орнатылды. Эрез-Розен метрикасы Эйнштейн теңдеулерінің дәл шешімі болып саналады және статикалық, осьтік симметриялы астрофизикалық нысандардың сыртқы гравитациялық өрісін сипаттайды, ал Хартл-Торн метрикасы Эйнштейн теңдеулерінің жуық шешімі болып есептеледі, ол баяу айналатын және аздап деформацияланған астрофизикалық нысандардың ішкі және сыртқы гравитациялық өрісін сипаттайды. Мақалада алға қойылған мақсатқа жету үшін Хартл-Торнның сыртқы метрикасы тек статикалық жағдайда қарастырылды.

Жоғарыда көрсетілген шешімдер арасында қатынас орнату үшін олар бірдей жуықтау және айналу болмағанда, дәлірек айтқанда, сызықтық квадрупольдік момент Q және толық масса квадраты M^2 жуықтауында қарастырылды. Хартл-Торн метрика координаталарында Эрез-Розен метрикасын жазу үшін алдымен Героч-Хансен мультипольдік моменттері есептелді, инвариантты моменттерді анықтау екі метрика параметрлері арасында байланыс орнатуға мүмкіндік берді. Содан кейін Эрез-Розен метрикасы Хартл-Торн метрикасының параметрлері Q, M арқылы жазылды және оның жуық өрнегі ~ Q және ~ M^2 шегінде алынды.

Әрі қарай, f_1 және f_2 екі белгісіз функциялары бар жалпы түрдегі координаталық түрлендіруді қолдану арқылы қарастырылған жуықтауда Эрез-Розен метрикасы Хартл-Торн метрикасының координатасында жазылды. Осыдан кейін екі шешімнің метрикалық тензорының радиалды және азимуталды құраушыларын теңестіре отырып, f_1 және f_2 функциялары айқындалды. Осылайша Эрез-Розен мен Хартл-Торн шешімі арасындағы байланысты анықтайтын координаталық түрлендірудің жуық өрнегі анықталды.

Мақалада қолданылған $\sim Q$ және $\sim M^2$ жуықтауы физикалық және постньютондық физикада аспан механикасының көптеген мәселелерін шешуге қолайлы екенін атап өткен жөн. Бұл жуықтау Зипой-Вурхис түрлендіруіне жүгінбеуге мүмкіндік береді. Себебі Зипой-Вурхис түрлендіруі $\sim Q$ жуықтауында, яғни, басқа ешқандай жуықтау ескерілмеген кезде қажетті қатаң математикалық талап болып саналады. Бұл координаталық түрлендірудің айқын түрі толығымен әрбір нақты жағдайда қолданылатын жуықтауға байланысты болады дегенді білдіреді.

Қорытындылай келе, Эрез-Розен және Хартл-Торн метрикалары $\sim Q$ және $\sim M^2$ жуықтауында статикалық жағдай үшін ұйытқу теориясының әдістерін қолдану негізінде зерттеу жасалды. Эрез-Розеннің сызықтық элементінің қарастырылған жуықтауда Хартл-Торн шешімімен сәйкес келетіні көрсетілді.

Осылайша жұмыста алынған нәтижелер әдебиетте алынған белгілі нәтижелермен сәйкес келеді және оларды әртүрлі астрофизикалық мәселелерге қолдануға болады. Мақала тек қана ғылыми емес, сонымен бірге академиялық мақсаттарды да көздейді және жалпы салыстырмалылық теориясы, аспан механикасы және релятивистік астрофизика бойынша арнайы курстарда көмекші және қосымша құрал ретінде пайдалануға болады.

Түйін сөздер: гравитациялық өріс үшін Эйнштейн теңдеулерінің дәл және жуық шешімдері, Эрез-Розен метрикасы, Хартл-Торн метрикасы, координаттық түрлендіру, квадрупольдік момент, Героч-Хансен мультипольдік моменттері, ұйытқу теориясының әдістері.

К. Бошкаев[1,2], А. Малыбаев[1], Э. Кеведо[1,3,4],
Г. Нурбакыт[1], А. Таукенова[1,2], А. Уразалина[1,2]

[1]КазНУ им. аль-Фараби, физико-технический факультет, Алматы, Казахстан;
[2]Национальная нанотехнологическая лаборатория открытого типа, Алматы, Казахстан;
[3]Римский университет Ла Сапиенца, физический факультет, Рим, Италия;
[4]Мексиканский национальный автономный университет, институт ядерных наук Мехико, Мексика

СООТВЕТСТВИЕ МЕТРИК ЭРЕЗА-РОЗЕНА И ХАРТЛА-ТОРНА В ПРИБЛИЖЕНИИ $\sim Q$ И $\sim M^2$

Аннотация. В данной работе установлена взаимосвязь между внешними решениями уравнений Эйнштейна для гравитационного поля, а именно метрикой Эреза-Розена и метрикой Хартла-Торна. Метрика Эреза-Розена является точным решением уравнений Эйнштейна и описывает внешнее гравитационное поле статических, аксиально-симметричных астрофизических объектов, в то время как метрика Хартла-Торна является приближенным решением уравнений Эйнштейна, которое описывает как внешние, так и внутренние гравитационные поля медленно вращающихся и слегка деформированных астрофизических объектов. Для достижения цели статьи рассмотрена только внешняя метрика Хартла-Торна в статическом случае.

Чтобы установить взаимосвязь между перечисленными метриками, они были рассмотрены в одинаковом приближении и в отсутствие вращения, точнее, в пределе линейного квадрупольного момента Q и квадрата полной массы M^2. Чтобы записать метрику Эреза-Розена в координатах метрики Хартла-Торна, сначала были вычислены мультипольные моменты Героча-Хансена, которые позволяют установить взаимосвязь между параметрами обеих метрик посредством вычисления инвариантных моментов. Затем метрика Эреза-Розена была записана через параметры метрики Хартла-Торна Q, M и был получен её приближенный вид в пределе $\sim Q$ и $\sim M^2$.

Далее, используя координатные преобразования, записанные в общем виде с двумя неизвестными функциями f_1 и f_2, метрика Эреза-Розена была записана в координатах метрики Хартла-Торна для рассматриваемого приближения. После этого, приравняв радиальные и азимутальные компоненты метрических тензоров двух рассматриваемых решений, были найдены искомые функции f_1 и f_2. Таким образом, был найден приближенный вид координатных преобразований, которые определяют взаимосвязь между решениями Эреза-Розена и Хартла-Торна.

Следует заметить, что приближение $\sim Q$ и $\sim M^2$, которое было использовано на протяжении всей статьи, является физическим и подходящим для решения большинства задач небесной механики в постньютоновской физике. Данное приближение позволяет не прибегать к преобразованию Зипоя-Вурхиса, которое является необходимым строгим математическим требованием в приближении $\sim Q$, т.е. когда не учитываются другие приближения. Это означает, что явная форма преобразования координат полностью зависит от приближения, которое используется в каждом конкретном случае.

В завершение следует отметить, что метрики Эреза-Розена и Хартла-Торна были исследованы для статического случая в приближении $\sim Q$ и $\sim M^2$ путем использования методов теории возмущений. Было показано, что приближенный линейный элемент Эреза-Розена совпадает с решением Хартла-Торна в рассматриваемом пределе.

Таким образом, результаты, полученные в этой статье, находятся в хорошем согласии с известными результатами из литературных источников и могут быть применены к различным астрофизическим задачам. Статья преследует не только чисто научные, но и академические цели, так как может быть использована в качестве вспомогательного и дополнительного материала для специальных курсов по общей теории относительности, небесной механике и релятивистской астрофизике.

Ключевые слова: точные и приближенные решения уравнений Эйнштейна для гравитационного поля, метрика Эреза-Розена, метрика Хартла-Торна, координатные преобразования, квадрупольный момент, мультипольные моменты Героча-Хансена, методы теории возмущений.

Information about authors:

Boshkayev Kuantay, Al-Farabi Kazakh National University, Almaty, Kazakhstan, PhD, Associate Professor, kuantay@mail.ru, http://orcid.org/0000-0002-1385-270X;

Malybayev Algis, Al-Farabi Kazakh National University, Almaty, Kazakhstan, PhD-student, algis_malybayev@mail.ru, https://orcid.org/0000-0003-2060-105X;

Quevedo Hernando, Universidad Nacional Autonoma de Mexico, Mexico, Mexico, PhD, Professor, quevedo@nucleares.unam.mx, http://orcid.org/0000-0003-4433-5550;

Nurbakyt Gulmira, Al-Farabi Kazakh National University, Almaty, Kazakhstan, PhD-student, gumi-nur@mail.ru, https://orcid.org/0000-0002-3275-2712;

Taukenova Aliya, Al-Farabi Kazakh National University, Almaty, Kazakhstan, PhD, Senior Lecturer, aliya_tauken@mail.ru, https://orcid.org/0000-0002-5154-330X;

Urazalina Aynur, Al-Farabi Kazakh National University, Almaty, Kazakhstan, PhD, Senior Lecturer, y.a.a.707@mail.ru, https://orcid.org/0000-0002-4633-9558

REFERENCES

[1] Stephani, H.; Kramer, D.; MacCallum, M.; Hoenselaers, C.; Herlt, E. Exact solutions of Einstein's field equations; Cambridge, UK: Cambridge University Press, 2003.

[2] Boshkayev, K.; Quevedo, H.; Ruffini, R. Gravitational field of compact objects in general relativity. Phys. Rev. D **2012**, 86, 064043, [arXiv:gr-qc/1207.3043]. doi:10.1103/PhysRevD.86.064043.

[3] Erez, G.; Rosen, N. The gravitational field of a particle possessing a quadripole moment. Bull. Res. Counc. Israel **1959**, 8F, 47.

[4] Hartle, J.B. Slowly Rotating Relativistic Stars. I. Equations of Structure. Astrophys. J. **1967**, 150, 1005. doi:10.1086/149400.

[5] Hartle, J.B.; Thorne, K.S. Slowly Rotating Relativistic Stars. II. Models for Neutron Stars and Supermassive Stars. Astrophys. J. **1968**, 153, 807. doi:10.1086/149707.

[6] Weyl, H. Zur Gravitationstheorie. Annalen der Physik **1917**, 359, 117–145. doi:10.1002/andp.19173591804.

[7] Doroshkevich, A.G.; Zel'Dovich, Y.B.; Novikov, I.D. Gravitational Collapse of Nonsymmetric and Rotating Masses. Soviet Journal of Experimental and Theoretical Physics **1966**, 22, 122.

[8] Winicour, J.; Janis, A.I.; Newman, E.T. Static, Axially Symmetric Point Horizons. Physical Review **1968**, 176, 1507–1513. doi:10.1103/PhysRev.176.1507.

[9] Young, J.H.; Coulter, C.A. Exact Metric for a Nonrotating Mass with a Quadrupole Moment. Physical Review **1969**, 184, 1313–1315. doi:10.1103/PhysRev.184.1313.

[10] Zeldovich, Y.B.; Novikov, I.D. Relativistic astrophysics. Vol.1: Stars and relativity; Chicago: University of Chicago Press, 1971, 1971.

[11] Quevedo, H.; Parkes, L. Geodesics in the Erez-Rosen space-time. General Relativity and Gravitation **1989**, 21, 1047–1072. doi:10.1007/BF00774088.

[12] Quevedo, H.; Mashhoon, B. Exterior gravitational field of a rotating deformed mass. Physics Letters A **1985**, 109, 13–18. doi:10.1016/0375-9601(85)90381-0.

[13] Quevedo, H. General static axisymmetric solution of Einstein's vacuum field equations in prolate spheroidal coordinates. Phys. Rev. D **1989**, 39, 2904–2911. doi:10.1103/PhysRevD.39.2904.

[14] Quevedo, H. Multipole Moments in General Relativity Static and Stationary Vacuum Solutions. Fortschritte der Physik **1990**, 38, 733–840. doi:10.1002/prop.2190381002.

[15] Quevedo, H.; Mashhoon, B. Exterior gravitational field of a charged rotating mass with arbitrary quadrupole moment. Physics Letters A **1990**, 148, 149–153. doi:10.1016/0375-9601(90)90770-O.

[16] Stergioulas, N. Rotating Stars in Relativity. Living Reviews in Relativity **2003**, 6, 3, [gr-qc/0302034]. doi:10.12942/lrr-2003-3.

[17] Boshkayev, K.; Rueda, J.A.; Ruffini, R.; Siutsou, I. On General Relativistic Uniformly Rotating White Dwarfs. Astrophys. J. **2013**, 762, 117, [arXiv:astro-ph.SR/1204.2070]. doi:10.1088/0004-637X/762/2/117.

[18] Belvedere, R.; Boshkayev, K.; Rueda, J.A.; Ruffini, R. Uniformly rotating neutron stars in the global and local charge neutrality cases. Nuclear Physics A **2014**, 921, 33–59, [arXiv:astro-ph.SR/1307.2836]. doi:10.1016/j.nuclphysa.2013.11.001.

[19] Weber, F.; Glendenning, N.K. Application of the improved Hartle method for the construction of general relativistic rotating neutron star models. Astrophys. J. **1992**, 390, 541–549. doi:10.1086/171304.

[20] Urbanec, M.; Miller, J.C.; Stuchlík, Z. Quadrupole moments of rotating neutron stars and strange stars. Mon. Not. Roy. Astr. Soc. **2013**, 433, 1903–1909, [arXiv:astro-ph.SR/1301.5925]. doi:10.1093/mnras/stt858.

[21] Zhami, B.A.; Boshkayev, K.A.; Quevedo, H.; Kalymova, Z.A. On some effects in the structure of white dwarfs. News of the National Academy of Sciences of the Republic of Kazakhstan. Physical and Mathematical Series **2019**, 6, 80–89. doi:10.32014/2019.2518-1726.75.

[22] Yagi, K.; Kyutoku, K.; Pappas, G.; Yunes, N.; Apostolatos, T.A. Effective no-hair relations for neutron stars and quark stars: Relativistic results. Phys. Rev. D **2014**, 89, 124013, [arXiv:gr-qc/1403.6243]. doi:10.1103/PhysRevD.89.124013.

[23] Mashhoon, B.; Theiss, D.S. Relativistic lunar theory. Nuovo Cimento B Serie **1991**, 106, 545–571. doi:10.1007/BF02726789.

[24] Frutos-Alfaro, F.; Soffel, M. On relativistic multipole moments of stationary space-times. Royal Society Open Science **2018**, 5, 180640, [arXiv:gr-qc/1606.07173]. doi:10.1098/rsos.180640.

[25] Boshkayev, K.; Quevedo, H.; Nurbakyt, G.; Malybayev, A.; Urazalina, A. The Erez–Rosen Solution Versus the Hartle–Thorne Solution. Symmetry **2019**, 11, 1324. doi:10.3390/sym11101324.

[26] Quevedo, H.; Toktarbay, S.; Aimuratov, Y. Quadrupolar gravitational fields described by the q-metric. arXiv e-prints **2013**, [arXiv:gr-qc/1310.5339].

[27] Quevedo, H. Multipolar Solutions. "Proceedings of the XIV Brazilian School of Cosmology and Gravitation, 2012, [arXiv:gr-qc/1201.1608].

[28] Geroch, R. Multipole Moments. II. Curved Space. Journal of Mathematical Physics **1970**, 11, 2580–2588, [https://doi.org/10.1063/1.1665427]. doi:10.1063/1.1665427.

[29] Fodor, G.; Hoenselaers, C.; Perjés, Z. Multipole moments of axisymmetric systems in relativity. Journal of Mathematical Physics **1989**, 30, 2252–2257, [https://doi.org/10.1063/1.528551]. doi:10.1063/1.528551.

[30] Ernst, F.J. New Formulation of the Axially Symmetric Gravitational Field Problem. Physical Review **1968**, 167, 1175–1177. doi:10.1103/PhysRev.167.1175.

[31] Ernst, F.J. New Formulation of the Axially Symmetric Gravitational Field Problem. II. Phys. Rev. **1968**, 168, 1415–1417. doi:10.1103/PhysRev.168.1415.

[32] Frutos-Alfaro, F.; Soffel, M. On the post-linear quadrupole-quadrupole metric. Revista de Matematica: Teoria y Aplicaciones **2017**, 24, 239, [arXiv:gr-qc/1507.04264]. doi:10.15517/rmta.v24i2.29856.

[33] Boshkayev, K.A.; Quevedo, H.; Abutalip, M.S.; Kalymova, Z.A.; Suleymanova, S.S. Geodesics in the field of a rotating deformed gravitational source. International Journal of Modern Physics A **2016**, 31, 1641006, [arXiv:gr-qc/1510.02016]. doi:10.1142/S0217751X16410062.

[34] Boshkayev, K.; Gasperín, E.; Gutiérrez-Piñeres, A.C.; Quevedo, H.; Toktarbay, S. Motion of test particles in the field of a naked singularity. Phys. Rev. D **2016**, 93, 024024, [arXiv:gr-qc/1509.03827]. doi:10.1103/PhysRevD.93.024024.

[35] Allahyari, A.; Firouzjahi, H.; Mashhoon, B. Quasinormal modes of a black hole with quadrupole moment. Phys. Rev. D **2019**, 99, 044005, [arXiv:gr-qc/1812.03376]. doi:10.1103/PhysRevD.99.044005.

[36] Allahyari, A.; Firouzjahi, H.; Mashhoon, B. Quasinormal Modes of Generalized Black Holes: delta-Kerr Spacetime. arXiv e-prints **2019**, [arXiv:gr-qc/1908.10813].