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**M.N. Kalimoldayev, A.T. Nurtazin, Z.G. Khisamiyev**

Institute of Information and Computational Technologies, Almaty, Kazakhstan.
E-mail: AbyzNutazin@mail.ru

# TWO PROPERTIES OF EXISTENTIALLY CLOSED COMPANIONS OF STRONGLY MINIMAL STRUCTURES

**Abstract.** The proposed article studies some properties of existentially closed companions of strongly minimal structures. A criterion for the existential closedness of an arbitrary strongly minimal structure is found in the article and it is proved that the existentially closed companion of any strongly minimal structure is itself strongly minimal. It also follows from the resulting description that all existentially closed companions of a given strongly minimal structure form an axiomatizable class whose elementary theory is complete and model-complete and, therefore, coincides with its inductive and forcing companions.

This is the reason for the importance of the work done and the high international significance of the theorems obtained in it. Another equally important consequence of this research is the discovery of an important subclass of strongly minimal theories. It should be noted that a complete description of this class of theories is an independent and extremely important task.

It is known that natural numbers with the following relation are an example of a strongly minimal structure in which the existential type of zero is not minimal. Then the method used in the proof of the last theorem shows that the existentially closed companion of this structure are integers with the following relation.

**Keywords:** Existentially closed companions, strongly minimal structures, forcing companions.

**1 Introduction.** The theory of existential closedness appeared in the middle of the twentieth century in the works of one of the recognized classics of model theory, Abraham Robinson [1], [2], as well as in [3] – [6]. It is currently one of the two most significant and developed areas of modern model theory. In the previous works of Nurtazin A.T. [7] - [10], the most basic form of the concept of companion theory, widely known in the theory of existential closure, is introduced and studied. In [7], [8], a criterion for the countable categoricity of this companion theory was found and some properties of existentially closed and forcing companions were studied.

Informally, any companion of a given structure is constructed from the same end structures as the original structure. It is known that all companions of this structure form an axiomatizable class, and all existentially closed ones in this class are contained in it [10]. It is natural to assume that a companion class containing some strongly minimal structure is quite simple and has a number of additional properties. In [11], statements about companion classes containing a strongly minimal structure were formulated. This paper describes some properties of companion completions of an arbitrary strongly minimal theory and provides proofs of the statements from [10].

Over the past fifty years in model theory, starting with the work of Michael Morley [12] and continuing in the works of Shelach [13] and Laskar and Poizat [14], all the most significant results are associated with their classification by degrees of stability that arose during this time. Moreover, totally transcendent (or ω-stable) theories were recognized as the most convenient for study. Recall that in these theories, all formulas have ranks and degrees, which respectively are countable ordinals and natural numbers. At the same time, all the successes in the study of these theories are primarily associated with the possibility of using the transfinite induction method in evidence. Historically, the initial impulse in the development of this direction was Los's hypothesis about uncountably categorical theories. At the same time, the core in the class of uncountably categorical theories turned out to have a single rank and degree, the so-called "strongly minimal theories." This paper studies existentially closed companions of strongly minimal theories.

In the work, the criterion of existential closedness of an arbitrary strongly minimal theory was found. It turned out that in this case the existentially closed companion of such a structure is axiomatized and coincides with its inductive and forcing companions. And its elementary theory is complete and model-complete and, therefore, $\forall\exists$- is axiomatizable.

Strongly minimal theories are basic in the study of an important and one of the most studied classes of theories at present - the theory of uncountable categoricity. Meanwhile, the available literature completely lacks both estimates of their quantum complexity and estimates of the complexity of the relations defined in them. The study proposed here shows that the concept of "companion"proposed in [7], [8], [9] can be successfully applied to solve these issues.

**1. Existential closedness criterion of a strongly minimal structure**

In this section, we will find conditions of existential closedness of an arbitrary strongly minimal structure.

THEOREM 1. For an arbitrary strongly minimal structure, the following three conditions are equivalent:

1) $M$ is existentially closed.

2) The existential type of any of its tuples is maximal.

3) The elementary theory $T = Th(M)$ is model complete.

Proof.

$1 \to 2$. By the existential closedness criterion from [11], this structure is strongly minimal if and only if the existential types of all its tuples are maximal.

$2 \to 3$. Let the existential type of any tuple from $M$ be maximal. Then, according to the criterion from [12], the structure $M$ is existentially closed. Recall that any elementary substructure of a structure $M$, including a simple model $M_0$ of a theory $T$, is also existentially closed. To prove the model completeness of the theory $T$, it is sufficient to show that the usual inclusion $M < N$ between any two models of this theory is actually elementary. First, we note that the models $M$ and $N$ are isomorphic to some $M_i$ and $M_j$ from the elementary chain $M_0 < M_1 < ... < M_n < ...$ of all countable models of the theory $T$. An uncomplicated reasoning shows that simple inclusion $M_i < M_j$ determines the isomorphism of an algebraic closure of an empty subset of the structure $M_i$ to the algebraic closure of an empty subset of the structure $M_j$. If at the same time $a_1,...,a_i$ are algebraically independent in the structure $M_i$, then they are also algebraically independent in $M_j$ and can be extended to the basis $\{a_1,...,a_i,a_{i+1},...,a_j\}$ of this structure. This determines the elementary inclusion $M_i$ in $M_j$ and with precision to automorphisms of these structures coincides with the original inclusion $M_i < M_j$. This completes the proof of the model completeness of the theory $T$.

$3 \to 1$. It follows from the model completeness of the theory $T$ that any of its models (including $M$) is existentially closed.

The theorem is proved.

Informally, any companion of a given structure consists of the same finite substructures as the original structure. It is known that all companions of this structure form an axiomatizable class, and all existentially closed ones in this class are contained in it. It is natural to assume that a companion class containing some strongly minimal structure is quite simple and has a number of additional properties. We will show in the next section that the existentially closed companion of a strongly minimal structure is also strongly minimal and is model complete.

**2. An existentially closed companion of a strongly minimal structure**

Informally, any companion of a given structure consists of the same finite substructures as the original structure. It is known that all companions of this structure form an axiomatizable class, and all existentially closed in this class are contained in it. It is natural to assume that a companion class containing some strongly minimal structure is quite simple and has a number of additional properties. Here we show that the existentially closed companion of a companion containing a strongly minimal structure is itself strongly minimal.

For certainty, we assume that the given companion class $C$ contains a countable saturated strongly minimal structure $M$. It is known from [7] that in any existentially closed structure from a companion class $C$, the existential type of any tuple is maximal. By induction on the lexicographic order on pairs of natural numbers $(m,n)$ we describe a procedure that allows to construct a sequence $M_0^0 < M_1^0 < ... < M_n^0 < ... < M_0^m < M_1^m < ... < M_n^m < ...$, of isomorphic $M$, the union $N$ of which will be a strong minimal structure and an existentially closed companion $M$.

Let the strongly minimal structure $M = M_0^0$ is not existentially closed. Then it contains tuples $a^0, a^1, ..., a^n, ...$ , whose existential types $p_0(x^0), p_1(x^1), ..., p_n(x^n), ...$ are not maximal. Using the compactness theorem, it is easy to prove the existence of an embedding of a structure $M$ in itself such that the existential type $q_0(x^0)$ ), of the image of the tuple $a^0$ is maximal.

Continuing further, one can obtain an increasing chain of strongly minimal structures $M_0^0 < M_1^0 < ... < M_n^0 < ...$ , isomorphic to $M$, in the union $M_0^1$ of which the existential types $q_0(x^0), q_1(x^1), ..., q_n(x^n), ...$ of tuples $a^0, a^1, ..., a^n, ...$ are maximal. Next, the process continues similarly. Then, in the union $N = \bigcup M_n^m$ , the existential types of all tuples are maximal.

To prove the strong minimality of the structure $N$ we first note that in the process of constructing the structure $N$ one can include intermediate steps used in [8] when expanding a given countable existentially closed structure into a homogeneous one. Recall that in the countable homogeneous existentially closed structure constructed in this case, the correspondence between any two tuples of elements having the same existential type continues to an automorphism of the homogeneous structure itself. Since over any finite set, all subsets, except one, allocated by maximum existential types, are finite, then these subsets themselves are allocated over this subset by finite formulasAt the same time, all elements satisfying infinite existential formulas are translated over the finite subset by structure automorphisms chosen by us and, therefore, satisfy the same infinite formulas. This proves the strong minimality of the structure $N$ . Now we show that any two existentially closed companions of the same minimal structure are elementary equivalent. For this we assume that there is a pair of existentially closed companion structures $N_1$ and $N_2$ of the structure $M$ . However, without limiting generality, we can assume that the inclusion $N_1 < N_2$ is enabled. Just as in the proof of the Los-Sushko theorem, the pair under consideration can be easily completed to the $\omega$- chain $N_1 < N_2 < N_3 < . . . N_{2n} < N_{2n+1} < . . .$ , in which the subchains $N_1 < N_3 < . . . < N_{2n+1} < . . .$ and $N_2 < . . . N_{2n} < . . .$ are elementary. But then the union $N$ of the built chain turns out to be a common elementary extension for each of the structures $N_1$ and $N_2$ , and the inclusion is elementary. We formulate the result as a separate statement.

THEOREM 2. If a given companion class has a strongly minimal structure, then any of its existentially closed companions is also strongly minimal. A complete and model-complete strongly minimal theory.

**Conclusion.** The results obtained in this paper link the three concepts of "model completeness, strong minimality, and existential closedness" that are currently most studied in modern model theory. This is the reason for the importance of the work done and the high international significance of the theorems obtained in it. Another equally important consequence of this research is the discovery of an important subclass of strongly minimal theories, which are strongly minimal model complete theories. It should be noted that a complete description of this class of theories is an independent and extremely important task.

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**М.Н. Қалимолдаев, А.Т. Нуртазин, З. Г. Хисамиев**

Ақпараттық және есептеуіш технологиялар институты, Алматы, Қазақстан

## ЭКЗИСТЕНЦИАЛДЫ ТҰЙЫҚТАЛҒАН СЕРІСТЕСТЕРДІҢ АСА МИНИМАЛДЫ ҚҰРЫЛЫМЫНЫҢ ЕКІ ҚАСИЕТІ

**Аннотация.** Алпыс жылдан астам тарихы бар экзистенциалды тұйықталу теориясы қазіргі модельдер теориясының дамыған бөлімдерінің бірі болып саналады және онда жеке позиция алады. Қазіргі уақытта алгебра мен модельдер теориясының классикалық бөлімдеріне экзистенциалды тұйықталу теориясының әдістерін және француз логигі Фрезенің революциялық идеяларын және экзистенциалды тұйықталудың жаңа ыңғайлы өлшемдерін, модельдің толықтығы мен жағдайын іздеу – «мәжбүр болу – модель» маңызды саналады. Қазіргі уақытта экзистенциалды және позитивті экзистенциалды тұйықталу теориясын және мәжбүрлі және индуктивті серіктес сияқты маңызды ұғымдарды зерттеуді одан әрі дамытуға мүмкіндік туады.

Зерттеуде өте аз құрылымдардың тұйықталу серіктесінің кейбір қасиеттері қарастырылған. Сондай-ақ, кез-келген күшті минималды құрылымның экзистенциалды тұйықталу өлшемі анықталды және кез-келген күшті минималды құрылымның экзистенциалды тұйықталу серіктесінің өте аз екендігі дәлелденді. Сондай-ақ, алынған сипаттамадан белгілі бір минималды құрылымның барлық тұйықталған серіктестері аксиоматизацияланатын сынып құрайды, олардың элементарлық теориясы толық және модельдік болып саналады, сондықтан оның индуктивті және мәжбүрлі серіктестерімен сәйкес келеді. Бейресми түрде берілген құрылымның кез-келген серіктесі бастапқы құрылымдағыдай ақырғы құрылымдардан жасалады. Белгілі құрылымның барлық серіктесі аксиоматизацияланатын сыныпты құрайтыны белгілі және сыныптағының барлығы экзистенциалды түрде тұйық. Біршама минималды құрылымы бар серіктес-сынып жеткілікті қарапайым және бірқатар қосымша қасиеті бар деп болжауға болады.

Еркін минималды теорияның экзистенциалды тұйықталу өлшемі анықталды. Бұл жағдайда мұндай құрылымның тұйықталу серігі аксиоматизацияланатын және оның индуктивті және мәжбүрлі серіктерімен сәйкес келетіні айқындалды. Күшті минималды теориялар маңызды және ең көп зерттелген теориялардың бірі – шексіз категориялық теория болып саналады. Сонымен бірге, қолжетімді әдебиеттерде олардың кванторлық санының күрделілігін бағалау да, олардағы қатынастардың күрделілігін бағалау мүлде жоқ. Мұнда ұсынылған зерттеулер «серіктес» ұғымын осы мәселелерді шешуде сәтті қолдануға болатындығын көрсетеді.

Жалпы индуктивті және экзистенциалды тұйықталған серіктестер теориясын дамыту индуктивті теория, классикалық құрылымдар мен теорияларға арналған модельдерді зерттеумен қатар жүруі керек.

**Түйін сөздер:** экзистенциалды тұйықталған серіктес, аса минималды құрылым, форсинг-серіктес.

**М.Н. Калимолдаев, А.Т. Нуртазин, З. Хисамиев**

Институт информационных и вычислительных технологии, Алматы, Казахстан

## ДВА СВОЙСТВА ЭКЗИСТЕНЦИАЛЬНО ЗАМКНУТЫХ КОМПАНЬОНОВ СИЛЬНО МИНИМАЛЬНЫХ СТРУКТУР

**Аннотация.** Имеющая более чем шестидесятилетнюю историю теория экзистенциальной замкнутости, являясь одним из наиболее развитых разделов современной теории моделей, занимает в ней обособленное положение. В настоящее время актуальным становится поиск методов теории экзистенциальной замкнутости классическим разделам алгебры и теории моделей и использования для достижения этого революционных идей французского логика Фрэсе и найденные автором данной статьи А.Т.Нуртазиным новых удобных критериев экзистенциальной замкнутости, модельной полноты и условия – быть форсинг-моделью. В настоящее время становится возможным дальнейшее развитие теории экзистенциальной и позитивной экзистенциальной замкнутости и изучение важных сопутствующих понятий таких, как форсинг – и индуктивный компаньоны.

В исследовании рассматриваются некоторые свойства экзистенциально замкнутых компаньонов сильно минимальных структур. Также найден критерий экзистенциальной замкнутости произвольной сильно минимальной структуры и доказано, что экзистенциально замкнутый компаньон любой сильно минимальной структуры сам сильно минимален. Из полученного описания также следует, что все экзистенциально замкнутые компаньоны данной сильно минимальной структуры образуют аксиоматизируемый класс, элементарная теория которого полна и модельно полна и, следовательно, совпадает с её индуктивным и форсинг-компаньонами. Неформально любой компаньон данной структуры строится из тех же конечных структур, что и исходная структура. Известно, что все компаньоны данной структуры образуют аксиоматизируемый класс, а все экзистенциально замкнутые в этом классе содержатся в нём. Естественно предположить, что компаньон-класс, содержащий некоторую, сильно минимальную структуру достаточно прост и обладает рядом дополнительных свойств.

В работе найден критерий экзистенциальной замкнутости произвольной сильно минимальной теории. Оказалось, что в этом случае экзистенциально замкнутый компаньон такой структуры аксиоматизируем и совпадает с его индуктивным и форсинг-компаньонами. Сильно минимальные теории являются базисными при исследовании важного и одного из наиболее изученных в настоящее время класса теорий – теории несчётной категоричности. Между тем в имеющейся литературе совершенно отсутствуют как оценки их кванторной сложности, так и оценки сложности определяемых в них отношений. Предлагаемое здесь исследование показывает, что для решения этих вопросов можно успешно применить предложенное понятие «компаньон».

Развитие общей теории индуктивных и экзистенциональных замкнутых компаньонов должно сопровождаться исследованием индуктивных теорий и моделей для классических структур и теорий.

**Ключевые слова:** экзистенциально замкнутые компаньоны, сильно минимальная структура, форсинг-компаньон.

**Information about the authors:**

Kalimoldayev M.N., academician of NAS RK, Dr. Sci.(Phys.-Math.), Director General of the Institute of Information and Computational Technologies, Almaty, Kazakhstan. E-mail: mnk@mail.ru, http://orcid.org/0000-0003-0025-8880;

Nurtazin A.T., leading research scientist of the Institute of Information and Computational Technologies, Cand. Sc. (Phys.-Math.). Almaty, Kazakhstan. E-mail: AbyzNutazin@mail.ru, https://orcid.org/0000-0003-3514-6205;

Khisamiyev Z.G., leading research scientist of the Institute of Information and Computational Technologies, Cand. Sc. (Phys.-Math.). Almaty, Kazakhstan. E-mail: khisamievz@mail.ru, https://orcid.org/0000-0003-3815-7233

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