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Y.Nessipbayev[1,2,3], K.Tulenov[1,2]

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;
[2]Al-Farabi Kazakh National University, Almaty, Kazakhstan.
[3]International Information Technology University, Almaty, Kazakhstan.
E-mail: yerlan.nessipbayev@gmail.com, tulenov@math.kz

BOUNDEDNESS OF THE HILBERT TRANSFORM FROM ONE ORLICZ SPACE TO ANOTHER

Abstract. In this paper, we investigate boundedness of the Hilbert transform from one Orlicz space to another. D.W. Boyd gave examples (see [1]) showing that reflexivity of X is both unnecessary and insufficient for boundedness of the Hilbert transform from X to itself. This may be somewhat surprising, since the condition for L_p to be reflexive ($1 < p < \infty$) is the same as the condition for the Hilbert transform to be bounded from L_p to L_p. However, D.W.Boyd considered only the cases when the domain and the range of the Hilbert transform coincide. Since L_p spaces are examples of rearrangement-invariant (r.i.) Banach function spaces, we consider boundedness of the Hilbert transform from one reaarangement-ivariant Banach function space to another. To be precise, we generalise the Boyd's results allowing the domain of the Hilbert transorm to be a particular Orlicz space defined on (0,1), and the range is different from the domain an other Orlicz space defined on (0,1). Moreover, we also consider boundedness of the Hilbert transform from one Lorentz space on (0,1) (which is also rearrangement invariant) to another Lorentz space (0,1). In case when the domain of the Hilbert transform is a Lorentz space $\Lambda_{\phi,p}(\mathbb{R}_+)$ which coincides with its range, the problems was fully resolved by D.W.Boyd. He showed (see [1]) that uniform convexity of $\Lambda_{\phi,p}(\mathbb{R}_+)$ $(1 < p < \infty)$ is necessary and sufficient condition for boundedness of the Hilbert transform. However, for the most important case when $p = 1$ the result was proved recently [2, Theorem 4.2]. Moreover, applying the main theorem D.W.Boyd obtained the following [1]:

Let L_ϕ be an Orlicz space. Then $\mathcal{H} \in B(L_\phi, L_\phi)$ if and only if L_ϕ os reflexive.

Towards these goals we also investigate boundedness of the Calderón operator from one rearrangement-invariant Banach function space to another. Such questions have been attracting a great deal of attention for many years, in particular in connection with embeddings of Sobolev spaces. In the present paper we discuss such boundedness problems for classical operators of great interest in analysis and its applications, namely the Hilbert transform and the Calderón operator. The action of these operators on specific classes of function spaces has been extensively studied over the several decades. Classical results are available for example in connection with familiar function spaces. Besides the importance of these operators is very well known, and their properties have been deeply studied.Classical Lorentz spaces which originated in 1950s and have been occurring occasionally later became extremely fashionable in 1990s when the fundamental papers appeared.

In this paper we study the boundedeness of such classical operators on rearrangement-invariant spaces, a class of function spaces that includes for example all Lebesgue, Lorentz, Orlicz, Marcinkiewicz spaces and more. Our focus is mainly on boundedness of the Hilbert transform from one Orlicz space to another. We also give examples of particular rearrangement-invariant spaces on which the Hilbert transform acts boundedly.

Key words: Hilbert transform, Calderón operator, Rearrangement invariant Banach functon spaces, Orlicz spaces, Lorentz spaces, Marcinkiewicz spaces.

1 Introduction

The purpose of this paper is to determine Orlicz spaces such that the Hilbert transform defines a bounded linear operator from one Orlicz space to another. Besides the Orlicz space, we deal with other spaces having the property of rearrangement-invariance.

Let $\mathcal{H}$ be the classical (singular) Hilbert transform (for measurable functions on $\mathbb{R}$), given by the formula

$$(\mathcal{H}x)(t) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{x(s)}{t-s}ds.$$

If X is a Banach space, let $B(X,X)$ denote the space of bounded linear operators from X into itself. A classical result of M.Riesz states that $\mathcal{H} \in B(L_p, L_p)$ if and only if $1 < p < \infty$. The main result of the paper by D.W.Boyd [1] generalises this as follows.

Theorem 1. *Let X be a rearrangement-invariant space. Define the operator E_s for $0 < s < \infty$ by $(E_s f)(x) = f(sx), f \in X$. Denote the norm of E_s as a member of $B(X,X)$ by $h(s;X)$. Then, $\mathcal{H} \in B(X,X)$ if and only if*

$$sh(s;X) \to 0 \text{ as } s \to 0+, \text{and } h(s;X) \to 0 \text{ as } s \to \infty.$$

Using this result, D.W.Boyd gives examples showing that reflexivity of X is both unnecessary and insufficient in order that $\mathcal{H} \in B(X,X)$. This may be somewhat surprising, since the condition for L_p to be reflexive (i.e. $1 < p < \infty$) is the same as the condition for $\mathcal{H} \in B(L_p, L_p)$. However, $1 < p < \infty$ also ensures that L_p is uniformly convex, so D.W.Boyd [1] obtained the following result:

Let X be the Lorentz space $\Lambda(\phi,p)$,$1 < p < \infty$. Then $\mathcal{H} \in B(X,X)$ if and only if X is uniformly convex.

In [1], D.W. Boyd characterized Lorentz spaces $\Lambda_{\phi,p}(\mathbb{R}_+)$ in which the Hilbert transform $\mathcal{H}$ is bounded from $\Lambda_{\phi,p}(\mathbb{R}_+)$ into itself in the case when $1 < p < \infty$. However, for the most important case when $p = 1$ the result was proved recently [2, Theorem 4.2]. Moreover, applying the main theorem D.W.Boyd obtained the following [1]:

Let L_ϕ be an Orlicz space. Then $\mathcal{H} \in B(L_\phi, L_\phi)$ if and only if L_ϕ os reflexive.

While D.W.Boyd investigated the boundedness of the Hilbert transform acting from a rearrangement-invariant space into itself, we rather consider the boundedness of the Hilbert transform from one (particular) Orlicz space, defined on (0,1), to another, i.e. we provide the Orlicz functions G_1 and G_2 such that (see Corollary 10) the Hilbert transform

$$\mathcal{H}: L_{G_2}(0,1) \to L_{G_1}(0,1).$$

We also give examples of Lorentz spaces such that the Calderón operator

$$S: \Lambda_\phi(0,1) \to \Lambda_\varphi(0,1).$$

2 Preliminaries

Let (I,m) denote the measure space, where throughout this paper $I = (0,1)$, equipped with Lebesgue measure m. Let $L(I,m)$ be the space of all measurable real-valued functions on (0,1) equipped with Lebesgue measure m, i.e. functions which coincide almost everywhere are considered identical. Let $L(I,m)$ be the space of all measurable real-valued functions on $(0,1)$ equipped with Lebesgue measure m. Define $L_0(0,1)$ to be the subset of $L(0,1)$ which consists of all functions x such that $m(\{t: |x(t)| > s\})$ is finite for some $s > 0$.

2.1 Rearrangement invariant Banach Function Spaces

Definition 2. *[3, Definition I. 1.1, p. 2] A mapping $\rho: L(I)^+ \to [0,\infty]$ is called a Banach function norm if, for all x, y, x_n, $(n = 1,2,3,\dots)$, in $L(I)^+$, for all m-measurable subsets Δ of $\mathbb{R}$, the following properties hold:*

1. ρ is a norm
2. $0 \le y \le x$ a.e. $\Rightarrow \rho(y) \le \rho(x)$
3. $0 \le x_n \uparrow x$ a.e. $\Rightarrow \rho(x_n) \uparrow \rho(x)$
4. $\rho(\Delta) < \infty \Rightarrow \rho(\chi_\Delta) < \infty$
5. $\rho(\Delta) < \infty \Rightarrow \int_\Delta xdm \le c_\Delta \rho(x)$

for some constant c_Δ, $0 < c_\Delta < \infty$, depending on Δ and ρ but independent of x.

Let ρ be a function norm. The set $E = E(\rho)$ of all functions x in $L(I)$ for which $\rho(|x|) < \infty$ is called a Banach function space. For any $x \in E$, define

$$\| x \|_E = \rho(|x|).$$

Define $L_0(I)$ to be the subset of $L(I)$ which consists of all functions x such that $m(\{t: |x(t)| > s\})$ is finite for some $s > 0$. Two functions x and y are called equimeasurable, if

$$m(\{t: |x(t)| > s\}) = m(\{t: |y(t)| > s\}).$$

For $x \in L_0(I)$, we denote by $\mu(x)$ the decreasing rearrangement of the function $|x|$. That is,

$$\mu(t,x) = \inf\{s \geq 0: m(\{|x| > s\}) \leq t\}, t > 0.$$

Definition 3. *[3, Definition 4.1, p. 59] A Banach function space E is called rearrangement-invariant if whenever x belongs to E and y is equimeasurable with x, then y also belongs to E and* $\| y \|_E = \| x \|_E$.

For the general theory of rearrangement invariant Banach function spaces, we refer the reader to [3, 4, 5, 6].

2.2 Köthe dual of Rearrangement invariant Function spaces

Next we define the Köthe dual space of rearrangement invariant Banach function spaces. Given a rearrangement invariant Banach function space E on (0,1), equipped with Lebesgue measure m, the Köthe dual space, denoted by $E^{\times}(0,1)$, is defined by

$$E(0,1)^{\times} = \left\{y \in S(0,1): \int_0^1 |x(t)y(t)|dt < \infty, \forall x \in E(0,1)\right\}.$$

$E^{\times}$ is a Banach space with the norm

$$\| y \|_{E(0,1)^{\times}} := \sup\left\{\int_0^1 |x(t)y(t)|dt: x \in E(0,1), \| x \|_{E(I)} \leq 1\right\}. \tag{2.1}$$

If E is a rearrangement invariant Banach function space, then $(E^{\times}, \|\cdot\|_{E^{\times}})$ is also a rearrangement invariant Banach function space (see [3, Section 2.4]). For more details on Köthe duality see [3, 5].

2.3 Lorentz and Marcinkiewicz spaces

Definition 4. *[4, Definition II. 1.1, p. 49] A function φ on the interval* [0,1] *is said to be quasiconcave if*

1. $\varphi(t) = 0 \Leftrightarrow t = 0$;
2. $\varphi(t)$ is positive and increasing for $t > 0$;
3. $\frac{\varphi(t)}{t}$ is decreasing for $t > 0$.

Observe that every nonnegative concave function on $[0,\infty)$ that vanishes only at origin is quasiconcave. The reverse, however, is not always true. But, we may replace, if necessary, a quasiconcave function φ by its least concave majorant $\tilde{\varphi}$ such that

$$\frac{1}{2}\tilde{\varphi} \leq \varphi \leq \tilde{\varphi}$$

(see [3, Proposition 5.10, p. 71]).

Let Ω denote the set of increasing concave functions $\varphi: [0,1] \to [0,1]$ for which $\lim_{t\to 0+}\varphi(t) = 0$ (or simply $\varphi(0+) = 0$). For a function φ in Ω, the Lorentz space $\Lambda_\varphi(0,1)$ is defined by setting

$$\Lambda_\varphi(0,1) := \left\{x \in L_0(0,1): \int_0^1 \mu(s,x)d\varphi(s) < \infty\right\}$$

equipped with the norm

$$\| x \|_{\Lambda_\varphi(0,1)} := \int_0^1 \mu(s,x)d\varphi(s). \tag{2.2}$$

The Lorentz spaces $(\Lambda_\varphi(0,1), \|\cdot\|_{\Lambda_\varphi(0,1)})$ are examples of rearrangement invariant Banach function spaces. For more details on Lorentz spaces, we refer the reader to [3, Chapter II.5] and [4, Chapter II.5]. Let ψ be a quasiconcave function on $(0,1)$. The space

$$M_\psi(0,1) = \{f \in L_1 : \| f \|_{M_\psi} < \infty\}$$

equipped with the norm

$$\| f \|_{M_\psi(0,1)} = \sup_{t\in(0,1)} \frac{1}{\psi(t)} \cdot \int_0^t \mu(s,f)dm$$

is the rearrangement invariant space with the fundamental function $\varphi(t) = \frac{t}{\varphi(t)} \cdot \mathbf{1}_{(0,1)}(t)$. The space $(M_\psi, \|\cdot\|_{M_\psi})$ is called the Marcinkiewicz space.

2.4 Orlicz spaces

Definition 5. *An Orlicz function is a function* $G: [0,1] \to [0,1]$ *such that*

(1) $G(0) = 0$, $G(\lambda_1) > 0$ for some $\lambda_1 > 0$ and $G(\lambda_2) < \infty$ for some $\lambda_2 > 0$.

(2) G is increasing.

(3) G is convex: $G(\alpha\lambda_1 + (1-\alpha)\lambda_2) \le \alpha G(\lambda_1) + (1-\alpha)G(\lambda_2), 0 \le \alpha \le 1$.

(4) G is left-continuous.

In what follows, unless otherwise specified, we always denote by G an Orlicz function.

For every Orlicz function G, we define a functional

$$I_G(f) = \int_0^1 G(|f|)dm \in [0,\infty]$$

and set

$$||f||_{L_G} = \inf\{a > 0 : I_G\left(\frac{f}{a}\right) \le 1\} \in [0,\infty]$$

for every measurable function $f: [0,1] \to \mathbf{R}$. We put here $\inf\{\emptyset\} = \infty$. The set

$$L_G = \{f \in L_0 : ||f||_{L_G} < \infty\}$$

equipped with the norm $||\cdot||_{L_G}$ is the rearrangement invariant space. The space $(L_G, ||\cdot||_{L_G})$ is called the *Orlicz space*. Orlicz spaces which generalize Lebesgue's scale in a direction essentially different from Lorentz spaces, received much attention too, see for instance [7, 8, 9, 10, 11, 12, 13].

2.5 Calderón operator and Hilbert transform

Let $E(0,1)$ be a r.i. Banach function space. For a function $x \in E(0,1)$, define formally the operator S as follows

$$(Sx)(t) := \frac{1}{t}\int_0^t x(s)ds + \int_t^1 x(s)\frac{ds}{s}, t > 0. \tag{2.3}$$

The operator S is called the Calderón operator. It is obvious that S is a linear operator. If $0 < t_1 < t_2$, then

$$\min\left(1,\frac{s}{t_2}\right) \le \min\left(1,\frac{s}{t_1}\right) \le \frac{t_2}{t_1} \cdot \min\left(1,\frac{s}{t_2}\right), s > 0.$$

Let x be nonnegative function on [0,1]. It follows from the first of these inequalities that $(Sx)(t)$ is a decreasing function of t. The operator S is often applied to the decreasing rearrangement $\mu(x)$ of a function x defined on some other measure space. Since $S\mu(x)$ is non-increasing itself, it is easy to see that $\mu(S\mu(x)) = S\mu(x)$. Throughout this paper, we write $\mathcal{A} \lesssim \mathcal{B}$ if there is a constant $c_{abs} > 0$ such that $\mathcal{A} \le c_{abs}\mathcal{B}$. We write $\mathcal{A} \approx \mathcal{B}$ if both $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{A} \gtrsim \mathcal{B}$ hold, possibly with different constants.

If $x \in E(0,1)$, (E is rearrangement-invariant space) then the classical Hilbert transform $\mathcal{H}$ is defined by the principal-value integral

$$(\mathcal{H}x)(s) := p.v. \frac{1}{\pi}\int_0^1 \frac{x(\eta)}{s-\eta}d\eta. \tag{2.4}$$

(see, e.g. [3, Chapter III. 4]). Boundedness of such classical operators on rearrangement-invariant spaces have attracted attention of leading mathematicians in the field of r.i. spaces and non-commutative analysis, see for example [14, 15, 16, 17, 18, 19, 20].

3 Main results

3.1 Boundedness of the Calderón operator on Orlicz spaces

Let S be the Calderón operator acting on Lorentz spaces

$$S:\Lambda_{\phi}(0,1)\to\Lambda_{\psi}(0,1)$$

In the next lemma we find the Lorentz function φ_0 such that $S:\Lambda_{\varphi_0}(0,1)\to L_1(0,1)$ is bounded and $\Lambda_{\varphi_0}(0,1)$ is maximal.

Lemma 6. *If* $\varphi_0(t)=tlog\left(\frac{e}{t}\right)+t, 0<t<1,$ *then* $\Lambda_{\varphi_0}(0,1)$ *is the maximal among all rearrangement invariant spaces such that* $S:\Lambda_{\varphi_0}(0,1)\to L_1(0,1)$. (Here and throughout this paper log stands for the natural logarithm)

Proof.

$$\int_0^1(Sx)(t)dt=\int_0^1\frac{1}{t}\int_0^t x(s)dsdt+\int_0^1\int_t^1\frac{x(s)}{s}dsdt=$$

$$=\int_0^1 x(s)\int_s^1\frac{1}{t}dtds+\int_0^1\frac{x(s)}{s}\int_0^s dtds=$$

$$=\int_0^1 x(s)\left(\log\left(\frac{1}{s}\right)+1\right)ds.$$

Hence $\varphi_0'(t)=\log\left(\frac{e}{s}\right)$. Then, by integrating we obtain

$$\varphi_0(t)=t\log\left(\frac{e}{t}\right)+t.$$

Now, we show that $\Lambda_{\varphi_0}(0,1)$ is maximal among such rearrangement invariant spaces. Indeed, let

$$S:E(0,1)\to L_1(0,1),$$

where $E(0,1)$ is arbitrary rearrangement invariant space. Then,

$$||x||_{\Lambda_{\varphi_0}}=||Sx||_{L_1}\lesssim||x||_E. \blacksquare$$

Therefore, $E(0,1)\subset\Lambda_{\varphi_0}(0,1)$.

Lemma 7. *Let* $\varphi(t)=tlog\left(\frac{e}{t}\right), t\in(0,1)$ *and* $\phi(t)=tlog^2\left(\frac{e}{t}\right), t\in(0,1)$. *Then*

$$S\left(\frac{\varphi(t)}{t}\right)\lesssim\frac{\phi(t)}{t}, t>0.$$

Proof. Indeed, integrating by parts, we obtain

$$S\left(\frac{\varphi(t)}{t}\right)=\frac{1}{t}\int_0^t\log\left(\frac{e}{s}\right)ds+\int_t^1\frac{\log\left(\frac{e}{s}\right)}{s}ds=\frac{1}{t}\left(s\log\left(\frac{e}{s}\right)\right)\Big|_0^t+\int_0^t ds-\frac{\log^2\left(\frac{e}{s}\right)}{2}\Big|_t^1$$

$$=\log\left(\frac{e}{t}\right)+1-\frac{1}{2}+\frac{\log^2\left(\frac{e}{t}\right)}{2}=\log\left(\frac{e}{t}\right)+\frac{1}{2}\log^2\left(\frac{e}{t}\right)+\frac{1}{2}=$$

$$=\frac{t\log\left(\frac{e}{t}\right)+\frac{1}{2}t\log^2\left(\frac{e}{t}\right)+\frac{1}{2}t}{t}\leq\frac{t\log^2\left(\frac{e}{t}\right)+\frac{1}{2}t\log^2\left(\frac{e}{t}\right)+t\log^2\left(\frac{e}{t}\right)}{t}=\frac{5}{2}\frac{t\log^2\left(\frac{e}{t}\right)}{t} \blacksquare$$

Note that both $\varphi(t)$ and $\phi(t)$ are Lorentz weight functions for $t\in(0,1)$.

Lemma 8. *Let* $\varphi(t)=tlog\left(\frac{e}{t}\right), t\in(0,1)$ *and* $\phi(t)=tlog^2\left(\frac{e}{t}\right), t\in(0,1)$. *Then the Calderón operator*

$$S:\Lambda_{\phi}(0,1)\to\Lambda_{\varphi}(0,1)$$

is bounded. Moreover,

$$S:M_{\varphi}(0,1)\to M_{\phi}(0,1)$$

is also bounded.

Proof. First we show the boundedness of the Calderón operator from one Lorentz space to another. It was shown in [17, lemma 10] that if $\lim_{t\to 0}\log\left(\frac{1}{t}\right)\varphi(t)=0$ and $\lim_{t\to\infty}\frac{\varphi(t)}{t}=0$, then

$$S:\Lambda_{\phi}(\mathbb{R}_+)\to\Lambda_{\varphi}(\mathbb{R}_+)$$

if and only if

$$S\left(\frac{\varphi(t)}{t}\right)\lesssim\frac{\phi(t)}{t}, t>0.$$

Here we consider the boundedness of the Calderón operator

$$S:\Lambda_{\phi}(0,1)\to\Lambda_{\varphi}(0,1).$$

It is easy to see that the first condition in [17, lemma 10], $\lim_{t\to 0}\log\left(\frac{1}{t}\right)\varphi(t)=0$, is satisfied. The condition for a function $\phi(t)$ such that $S\left(\frac{\varphi(t)}{t}\right)\lesssim\frac{\phi(t)}{t}$ for all $t>0$ is satisfied by lemma 7. Hence, by [17, lemma 10] we obtain that the Calderón operator

$$S:\Lambda_{\phi}(0,1)\to\Lambda_{\varphi}(0,1)$$

is bounded. It is well known fact that the associate space (Köthe dual) of the Lorentz space Λ_{φ} with weight function φ is the Marcienkiewicz space M_{φ} and $||\cdot||_{\Lambda_{\varphi}^{\times}}=||\cdot||_{M_{\varphi}}$. [4, Theorem 5.2, p.112]

Hence,

$$S:M_{\varphi}(0,1)\to M_{\phi}(0,1)$$

is also bounded.

Theorem 9. *Let* $G_1(t)=e^{|t|^{\frac{1}{2}}}-\frac{|t|}{2}-|t|^{\frac{1}{2}}-1$ *and* $G_2(t)=e^{|t|}-1$. *Then the Calderón operator*

$$S:L_{G_2}(0,1)\to L_{G_1}(0,1)$$

is bounded.

Proof. It is known from [19, see Lemma 4.3] that $L_{N_p}=M_{\psi_p}$ (with norm equivalence) holds for every $p>0$, where

$$N_p(t):=e^{|t|^p}-\sum_{k=0}^{|\frac{1}{p}|}\frac{|t|^{kp}}{k!},p\in(0,1);$$

$$N_p(t):=e^{|t|^p}-1,p\geq 1,t\in\mathbf{R}.$$

and

$$\psi(t):=t\log^{1/p}\left(\frac{e}{t}\right),t\geq 0.$$

If we choose $p=\frac{1}{2}$, then $\psi(t)=t\log^2\left(\frac{e}{t}\right)$ coincides with the $\phi(t)$ above. The corresponding Orlicz function is $N_{\frac{1}{2}}(t)=e^{|t|^{\frac{1}{2}}}-\frac{|t|}{2}-|t|^{\frac{1}{2}}-1$. For convenience let us denote $N_{\frac{1}{2}}(t)=G_1(t)$. Then

$$M_{\phi}(0,1)=L_{G_1}(0,1).$$

Similarly, we need to find the Orlicz function G_2 such that $M_{\varphi}(0,1)=L_{G_2}(0,1)$. For $p=1$ we have $\psi(t)=t\log\left(\frac{e}{t}\right)$ which is equal to $\varphi(t)$. Hence, the corresponding Orlicz function is $N_1(t)=e^{|t|}-1$. Let us denote $N_1(t)=G_2(t)$ so that $M_{\varphi}(0,1)=L_{G_2}(0,1)$ holds. Thus, we have found the Orlicz functions G_1 and G_2 such that the Calderón operator

$$S:L_{G_2}(0,1)\to L_{G_1}(0,1)$$

is bounded.

3.2 Boundedness of the Hilbert transform from one Orlicz space to another

In this section we use the equivalence (up to equimeasurable functions) of the Calderón operator S and the Hilbert transform $\mathcal{H}$ [3, chapter 3.4].

Corollary 10. *The Hilbert transform*

$$\mathcal{H}:L_{G_2}(0,1)\to L_{G_1}(0,1),$$

where $G_2(t)=e^{|t|}-1$ and $G_1(t)=e^{|t|^{\frac{1}{2}}}-\frac{|t|}{2}-|t|^{\frac{1}{2}}-1$, is bounded.

Proof. An estimate of the Hilbert transform $(\mathcal{H}f)(t)$ from above by the function $S(\mu(t,f))$ is given in [3, Theorem III.4.8, p.138]. The theorem states that if $S(\mu(1,f))<\infty$, then the Hilbert transform $(\mathcal{H}f)(t)$ exists a.e. Furthermore,

$$\mu(t,\mathcal{H}f)\leq cS(\mu(t,f)),t>0$$

for some constant c independent of f and t. The corresponding lower estimate is false. However, there is the following substitute. If $S(\mu(1,f)) < \infty$, then there exists a function g equimeasurable with f such that

$$S(\mu(t,f)) \leq 2\pi\mu(t,\mathcal{H}g), t > 0.$$

[see [3, Proposition III.4.10, p. 140]].

Therefore, all results obtained in the previous section for the Calderón operator S will also remain valid for the Hilbert transform $\mathcal{H}$. In particular, we also obtain boundedness of the Hilbert transform from one Lorentz space to another and from one Marcienkiewicz space to another as well.

Е.Х.Несипбаев[1,2,3], К.С.Туленов[1,2]

[1]Институт математики и математического моделирования,Алматы, Казахстан;
[2]Казахский национальный университет им. Аль-Фараби, Алматы, Казахстан;
[3]Международный университет информационных технологий, Алматы, Казахстан

ОГРАНИЧЕННОСТЬ ПРЕОБРАЗОВАНИЯ ГИЛЬБЕРТА ИЗ ОДНОГО ПРОСТРАНСТВА ОРЛИЧА В ДРУГОЕ

Аннотация. В этой статье исследуется ограниченность преобразования Гильберта из одного пространства Орлича в другое. Д. В. Бойд привел примеры (см. [1]), показывающие, что рефлексивность X не является как необходимым, так и достаточным условием для ограниченности преобразования Гильберта. Это может несколько удивить, поскольку условие рефлексивности L_p $(1 < p < \infty)$ совпадает с условием ограниченности преобразования Гильберта из L_p в L_p. Однако Д. В. Бойд рассматривал только случаи, когда область определения и область значений преобразования Гильберта совпадают. Поскольку пространства L_p являются примерами перестановочно-инвариантных банаховых функциональных пространств, мы рассматриваем ограниченность преобразования Гильберта из одного перестановочно-инвариантного банахова пространства функций в другое. Точнее, мы обобщаем результаты Бойда, позволяя областью определения преобразования Гильберта быть конкретным пространством Орлича, определенным в (0,1); а областью значений также быть пространством Орлича, определенным в (0,1), отличным от области определения. Кроме того, мы также рассматриваем ограниченность преобразования Гильберта из одного пространства Лоренца на (0,1) (которое также является инвариантно-перестановочным) в другое пространство Лоренца (0,1).

Для достижения этих целей мы также рассмотрим ограниченность оператора Кальдерона из одного перестановочно-инвариантного банахова пространства в другое. В случае, когда областью преобразования Гильберта является пространство Лоренца $\Lambda_{\phi,p}(\mathbb{R}_+)$, совпадающее с его областью значений, задачи были полностью решены Д. В. Бойдом. Он показал (см. [1]), что равномерная выпуклость $\Lambda_{\phi,p}(\mathbb{R}_+)$ $(1 < p < \infty)$ является необходимым и достаточным условием ограниченности преобразования Гильберта. Однако для наиболее важного случая, когда p = 1, результат был недавно доказан [2, теорема 4.2]. Кроме того, применяя основную теорему Д.Бойд, получили следующее [1]:

Пусть L_ϕ - пространство Орлича. Тогда $\mathcal{H} \in B(L_\phi, L_\phi)$ тогда и только тогда, когда L_ϕ является рефлексивным.

Такие вопросы привлекают большое внимание математиков на протяжении многих лет, в частности, в связи с вложениями соболевских пространств. В настоящей статье мы обсуждаем такие проблемы ограниченности для классических операторов, представляющих большой интерес для анализа и его приложений, а именно для преобразования Гильберта и оператора Кальдерона. Действие этих операторов на конкретные классы функциональных пространств широко изучалось в течение нескольких десятилетий. Классические результаты доступны, например, в связи со знакомыми функциональными пространствами. Кроме того, приложения этих операторов очень хорошо известны, и их свойства были глубоко изучены. Классические пространства Лоренца, которые возникли в 1950-х годах, стали чрезвычайно модными в 1990-х годах, когда появились фундаментальные статьи.

В этой статье мы изучаем ограниченность таких классических операторов на пространствах, инвариантных относительно перестановки, то есть класс функциональных пространств, который включает, например, все пространства Лебега, Лоренца, Орлича, Марцинкевича и многие другие. Основное внимание мы уделяем ограниченности преобразования Гильберта из одного пространства Орлича в другое. Мы также приводим примеры конкретных перестановочно-инвариантных пространств, в которых преобразование Гильберта действует ограниченно.

Ключевые слова: преобразование Гильберта, оператор Кальдерона, перестановочно-инвариантные банаховы пространства функций, пространства Орлича, пространства Лоренца, пространства Марцинкевича.

Е.Х.Несіпбаев[1,2,3], Қ.С.Төленов[1,2]

[1] Математика және математикалық модельдеу институты, Алматы, Қазақстан;
[2] Әл-Фараби атындагы Қазақ ұлттық университеті, Алматы, Қазақстан.
[3] Халықаралық ақпараттық технологиялар университеті, Алматы, Қазақстан;

ГИЛЬБЕРТ ТҮРЛЕНДІРУІНІҢ БІР ОРЛИЧ КЕҢІСТІГІНЕН ЕКІНШІСІНЕ ШЕНЕЛГЕНДІГІ

Аннотация. Бұл мақалада біз Гильберт түрлендіруінің бір Орлич кеңістігінен екінші Орлич кеңістігіне шенелгендігін зерттейміз. Д.В. Бойд өзінің жұмысында [1] Гильберт түрлендіруінің X кеңістігіндегі шенелгендігі үшін X кеңістігінің рефлексивтілігі қажет те жеткілікті де емес екендігін көрсетті. Бұл біршама таңқаларлық жағдай болуы мүмкін, өйткені L_p $(1 < p < \infty)$ рефлексті болады сонда тек сонда гана Гильберт түрлендіруі L_p-дан L_p-га шенелген оператор болса. Алайда, Д.В. Бойд Гилберт түрлендіруінің аңықталу облысы мен мәндер облысы бірдей кеңістік болатын жағдайларды толықтай зерттеді. L_p кеңістігі алмастырмалы-инвариантты функцияларының Банах кеңістігі болгандықтан, біз бұл жұмыста Гильберт түрлендіруінің аңықталу облысы мен мәндер облыстары әр түрлі болган жағдайды қарастырамыз. Атап айтқанда, Гилберт түрлендіруінің (0,1) аралыгында анықталған өлшемді функциялардың Орлич кеңістіктерінде шенелгендігін зерттейміз, ягни бір Орлич кеңістігінен басқа Орлич кеңістігіне шенелгендігін көрсетеміз. Сонымен қатар, біз Гильберт түрлендіруінің бір Лоренц кеңістігінен басқа Лоренц кеңістігіне шенелгендік критерийін қарастырамыз. Егер Гилберт түрлендіруінің аңықталу облысы Лоренц кеңістігі $\Lambda_{\phi,p}(\mathbb{R}_+)$ болса, және оның мәндер облысы да сол Лоренц кеңістігі $\Lambda_{\phi,p}(\mathbb{R}_+)$ $(1 < p < \infty)$ болса, онда осы проблеманы Д.В.Бойд толықтай шешкен. Ол ([1]) Лоренц кеңістігінің $\Lambda_{\phi,p}(\mathbb{R}_+)$ $(1 < p < \infty)$ біртекті дөңестілігі Гильберт түрлендіруінің шенелгендігі үшін қажетті және жеткілікті шарт екенін көрсетті. Алайда, ең маңызды жағдай үшін p = 1 болган кездегі нәтиже тек қана жақында дәлелденді [2, Теорема 4.2]. Сонымен қатар, Д.В.Бойд негізгі теоремасын қолдана отырып, төмендегі нәтиже алды [1]:

L_ϕ Orlicz кеңістігі болсын. Онда $\mathcal{H} \in B(L_\phi, L_\phi)$ сонда және тек қана сонда егер L_ϕ рефлексивті болса. Осы мақсаттарга жету үшін біз Кальдерон операторының бір банах алмастырмалы-инвариантты функцияларының Банах кеңістігінен екіншісіне бейнелейтіндігін қарастырамыз.

Мұндай сұрақтар көптеген жылдар бойы, әсіресе Соболев кеңістігін ендіруге байланысты көпшіліктің назарын аударды. Осы жұмыста біз талдауга және оны қолдануга үлкен қызығушылық тудыратын классикалық операторлар үшін, мысалы, Гильберт түрлендіруі және Кальдерон операторы үшін мұндай шенелгендік проблемаларын талқылаймыз. Бұл операторлардың функционалдық кеңістіктердің арнайы кластарындағы әрекеті бірнеше ондаған жылдар бойы кең зерттелген. Классикалық нәтижелерге, мысалы, таныс функциялар кеңістігіне байланысты қол жетімді. Сонымен қатар, осы операторлардың маңыздылыгы белгілі, және олардың қасиеттері терең зерттелген. 1950 жылдары пайда болған классикалық Лоренц кеңістігі 1990 жылдардан бастап негізгі зерттеулер пайда болган кезде өте көп қолданылады.

Бұл жұмыста біз осындай классикалық операторлардың шенелгендігін алмастырмалы-инвариантты кеңістіктерде, мысалы, барлық Лебег, Лоренц, Орлич, Марцинкевич кеңістіктерінде және басқа функционалдық кеңістіктерде зерттейміз. Біздің назарымыз, негізінен, Гильберт түрлендіруінің бір Орлич кеңістігінен екіншісіне шенелгендігінде. Сонымен қатар, біз Гилберт түрлендіруінің шектеулі болатындай инварианттық кеңістіктердің мысалдарын келтіреміз.

Түйін сөздер: Гильберт түрлендіруі, Кальдерон операторы, алмастырмалы-инвариантты функцияларының Банах кеңістігі, Орлич кеңістігі, Лоренц кеңістігі, Марцинкевич кеңістері.

Information about authors:

Nessipbayev Yerlan Khabdulkhanovich – senior lecturer at International Information technology University, department of Mathematical and Computer Modeling, PhD student at Al-Farabi Kazakh National University and Institute of Mathematics and Mathematical Modeling, yerlan.nessipbayev@gmail.com, https://orcid.org/0000-0002-0480-7296;

Tulenov Kanat Serikovich – senior research fellow at Institute of Mathematics and Mathematical Modeling, postdoc at Al-Farabi Kazakh National University, tulenov@math.kz, https://orcid.org/0000-0003-4104-7422

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