N E W S
OF THE NATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
PHYSICO-MATHEMATICAL SERIES
ISSN 1991-346X
Volume 1, Number 317 (2018), 34 – 50

UDC 517.956.32

M.I. Akylbayev[1], A. Beysebayeva[2], A. Sh. Shaldanbayev[2]

[1] Regional Social-Innovational University,Shymkent;
[2]M.Auezov South Kazakhstan State University, Shymkent.
musabek_kz@mail.ru akbope_a@mail.ru shaldanbaev51@mail.ru

ON THE PERIODIC SOLUTION OF THE GOURSAT PROBLEM FOR A WAVE EQUATION OF A SPECIAL FORM WITH VARIABLE COEFFICIENTS

Abstract. In this work the task of Goursat in a characteristic quadrangle for a wave equation of an express view with variable coefficients is solved. The spectral impression of the decision not traditional for such Voltaire tasks is gained. For this purpose as a vvvspomogoalny task the spectral task for the equation is used with we otklonyashchitsya by an argument. It is shown that the oprator of a type of Su(x)=u(1-x) plays a role of the operator Schmidt встречающиееся in decomposition of Voltaire operators.

Keywords: Volterra operators, indefinite metric, Goursat problem, similarity operators, spectrum, spectral decomposition, Fourier method, orthogonal basis, the Hulbert-Schmidt theorem.

1. Introduction. The investigations of the Dirichlet problem for the string vibration equation in a bounded region go back to J. Hadamard (Filler) who first noted the uniqueness of the solution in the rectangle. D.Burgin and Duffin [2] considered the Dirichlet problem for the equation $u_{xx} = u_{tt}$in the rectangle { 0 < x <X ; 0<t<T}. It is shown that the un uniqueness of a solution in a certain space appears if and only if X / T is rational. The existence theorems for a solution in classical spaces are established, and the smoothness of the solution is as greater as the smoothness is larger of the boundary function and as worse the number X / T is approximated to the rational numbers. Also the Neumann problem considered. Later these results were refined and generalized by various authors (see, for example, [3], [4], [5], [6]). Sobolev [7] constructed an example of a well-posed boundary value problem in a rectangle for a hyperbolic system of equations, Yu.M. Berezanskii [8] constructed a class of regions with angles, a change in the domain inside which leads to a continuous changing of solution of the Dirichlet problem. For regions with a smooth boundary in smooth spaces, only the question of the uniqueness of solution of the Dirichlet problem was studied (see, for example, the work of RA Aleksandryan [9]). In work [3], Arnol'd, applying his results on the maps of the circle into itself, refines the results of [2], indicating that the proof of theorems on the existence of classical solutions of the Dirichlet problem can be carried over to the case of an ellipse. Row of investigations T.Sh. Kalmenov and M.A. Sadybekov's are also devoted to boundary value problems of hyperbolic equations [10] - [12], the results of these researches are summarized in the monograph [13].

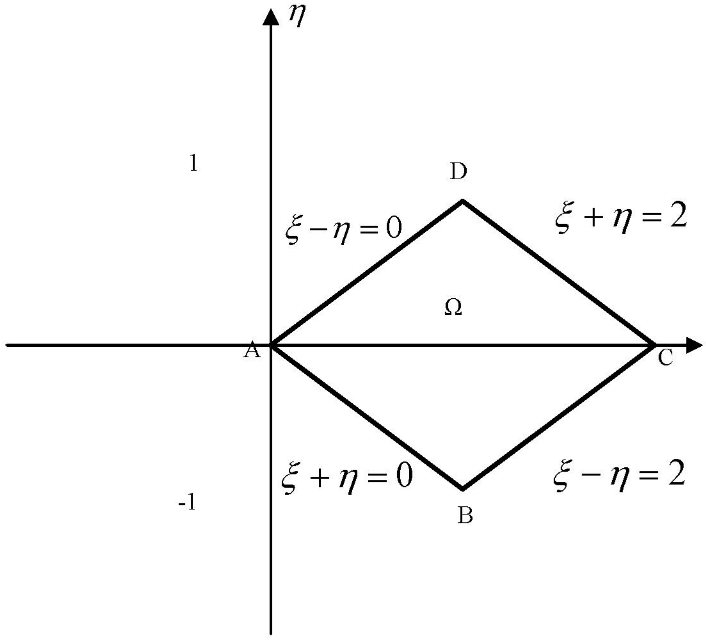
In [14], using the new general method, the properties of solutions of the Cauchy problem, are researched as well as of the first, second and third boundary value problems in the disk for a second-order hyperbolic equation with constant coefficients are investigated. The application of this method to higher-order equations can be found in [15]. A new and relatively simple method for constructing a system of polynomial solutions of the Dirichlet problem for second-order hyperbolic equations with constant coefficients in the disk is proposed in [16], and it is also proposed to construct a complete set of eigenfunctions for the Dirichlet problem for the string oscillation equation. The eigenfunctions constructed in this paper coincide with the eigenfunctions constructed earlier by RA Aleksandaryan [9].

The analysis of the contents of these studies has shown that the spectral properties of these boundary-value problems depend on the geometry of the region, in particular, on the group of motion of the region. A non-equilateral triangle does not have a symmetry group, so we abandoned the characteristic triangle and began to consider boundary value problems inside the characteristic quadrangle. In this case, equations with deviating arguments appear naturally, which deserve a separate investigation [17] - [24].

1. Let- Ω be the characteristic quadrangle of the wave equation,

$$u_{\xi\xi}-u_{\eta\eta}+q\left(\frac{\xi-\eta}{2}\right)\left(u_\xi+u_\eta\right)+p\left(\frac{\xi+\eta}{2}\right)\left(u_\xi-u_\eta\right)+p\left(\frac{\xi+\eta}{2}\right)q\left(\frac{\xi-\eta}{2}\right)u=f(\xi,\eta) \quad (1)$$

with the sides $AB: \xi+\eta=0, BC: \xi-\eta=2, CD: \xi+\eta=2, DA: \xi-\eta=0$
(see Pic.1).



Pic. 1

Suppose that the right-hand side of equation (1) is a periodic function with some periods. The question is whether equation (1) can have a periodic solution for the corresponding behavior of the coefficients. It is known that a periodic problem is poorly posed for the wave equation because of the presence of an infinite eigenvalue at the point λ = 0. Therefore, we consider the Goursat problem for equation (1) and study the possibilities of periodic continuability of the solution of this problem to the whole (ξ,η)plane.

Formulation of the problem. Find the periodic solution of the Goursat problem for the wave equation

$$\begin{cases} u_{\xi\xi} - u_{\eta\eta} + q\left(\frac{\xi-\eta}{2}\right)\left(u_\xi + u_\eta\right) + \rho\left(\frac{\xi+\eta}{2}\right)\left(u_\xi - u_\eta\right) + \rho\left(\frac{\xi+\eta}{2}\right) q\left(\frac{\xi-\eta}{2}\right) u = f(\xi,\eta) \; (1) \; (1) \\ u|_{AB} = 0, u|_{BC} = 0 (2) \;\; (2) \end{cases}$$

To solve this problem, we make a variables of change. Assuming,

$x = \frac{\xi+\eta}{2}, y = \frac{\xi-\eta}{2}$,we have

$$\xi = x + y, \eta = x - y;\; u(\xi,\eta) = u(x+y, x-y) = \hat{u}(x,y);$$

$$u_\xi = \hat{u}_x \cdot x_\xi + \hat{u}_y \cdot y_\xi = \hat{u}_x \cdot \frac{1}{2} + \hat{u}_y \cdot \frac{1}{2} = \frac{1}{2}(\hat{u}_x, \hat{u}_y);$$

$$u_{\xi\xi} = \frac{1}{2}\left[\hat{u}_{xx} \cdot x_\xi + \hat{u}_{xy} \cdot y_\xi + \hat{u}_{yx} \cdot x_\xi + \hat{u}_{yy} \cdot y_\xi\right] = \frac{1}{2}\left[\hat{u}_{xx} \cdot \frac{1}{2} + \hat{u}_{xy} \cdot \frac{1}{2} + \hat{u}_{yx} \cdot \frac{1}{2} + \hat{u}_{yy} \cdot \frac{1}{2}\right] =$$

$$= \frac{1}{4}\left[\hat{u}_{xx} + 2\hat{u}_{xy} + \hat{u}_{yy}\right];$$

$$u_\eta = \hat{u}_x \cdot x_\eta + \hat{u}_y \cdot y_\eta = \hat{u}_x \cdot \frac{1}{2} - \hat{u}_y \cdot \frac{1}{2} = \frac{1}{2}(\hat{u}_x, \hat{u}_y);$$

$$u_{\eta\eta} = \frac{1}{2}\left[\hat{u}_{xx} \cdot x_\eta + \hat{u}_{xy} \cdot y_\eta - \hat{u}_{yx} \cdot x_\eta - \hat{u}_{yy} \cdot y_\eta\right] = \frac{1}{2}\left[\hat{u}_{xx} \cdot \frac{1}{2} - \hat{u}_{xy} \cdot \frac{1}{2} - \hat{u}_{yx} \cdot \frac{1}{2} + \hat{u}_{yy} \cdot \frac{1}{2}\right] =$$

$$= \frac{1}{4}\left[\hat{u}_{xx} - 2\hat{u}_{xy} + \hat{u}_{yy}\right];$$

$$u_{\xi\xi} - u_{\eta\eta} = \hat{u}_{xy}.$$

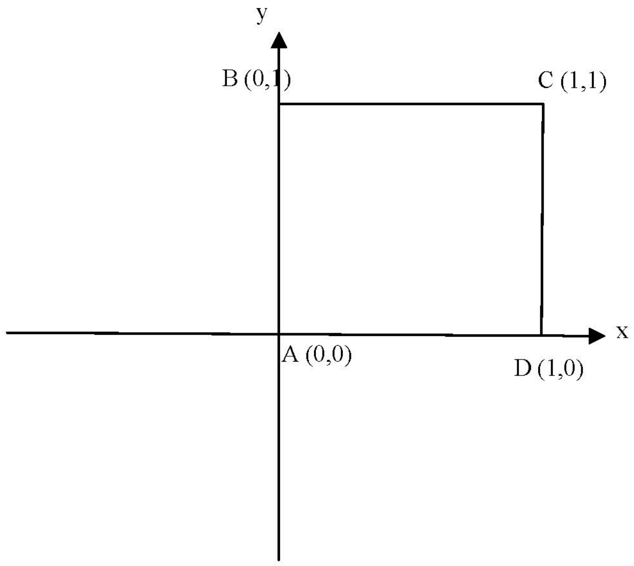
After the replacement, equation (1) takes the form

$$\hat{u}_{xy} + \hat{q}(y)\hat{u}_x + \hat{p}(x)\hat{u}_y + \hat{p}(x)\hat{q}(y)\hat{u}(x,y) = \hat{f}(x,y).$$

After releasing the caps after the transformation, we get

$$\left[\frac{\partial}{\partial x} + p(x)\right] \cdot \left[\frac{\partial}{\partial y} + q(x)\right] \cdot u(x,y) = f(x,y).$$

This is a wave equation of a special form. Now let us consider the boundary conditions, and the region of variation of the new variables *x*, *y*. The mapping $x = \frac{\xi+\eta}{2}, y = \frac{\xi-\eta}{2}$ maps the domain Ω to the domain D (see Pic.2).



Pic.2

Consequently, our original problem takes the following form

$$\begin{cases} \left[\frac{\partial}{\partial x} + p(x)\right] \cdot \left[\frac{\partial}{\partial y} + q(x)\right] \cdot u(x,y) = f(x,y), (x,y) \in D & (3) \\ u|_{x=0} = 0, u|_{y=1} = 0. & (4) \end{cases}$$

The aim of this paper is to solve the Goursat problem (3) - (4) using the methods of the spectral theory of differential equations with deviating arguments [17-24], and the proof of the periodicity of the obtained solution.

2. The researching methods

First we study the corresponding spectral problem:

$$\begin{cases} y'(x) + q(x)y(x) = \lambda \cdot y(1-x), x \in (0,1), & (5) \\ y(0) = 0. & (6) \end{cases}$$

where $q(x)$ –is a continuous function.

Question: under what conditions on $q(x)$ the operator of problem (5) - (6) is similar to the operator of problem

$$\begin{cases} y'(x) = \lambda \cdot y(1-x), x \in (0,1), & (5') \\ y(0) = 0. & (6') \end{cases}$$

Lemma 2.1. If$H=L_2(0,1)$ and

$$q(x) + q(1-x) = 0,$$

then the operators

$$A = \frac{d}{dx}, D(A) = \{y(x) \in W_2^1, y(0) = 0\}$$

$$B = \frac{d}{dx} + q(x), D(B) = \{z(x) \in W_2^1, z(0) = 0\}$$

are similar to each other.

Proof. We seek the transformation operator in the form

$$z(x) = Ty(x) = e^{\int_0^x q(t)dt} \cdot y(x).$$

Then we have $z(0) = e^0 \cdot y(0) = 0fory(x) \in D(A)$. Hence the operator T takes the domain of the operator A to the domain of the operator, that is, $T: D(A) \to D(B)$.

Further,

$$z'(x) = y'(x) \cdot e^{\int_0^x q(t)dt} + q(x) \cdot y(x) \cdot e^{\int_0^x q(t)dt} = T[y'(x) + q(x) \cdot y(x)] = TBy(x).$$

Consequently

$$Az = z'(x) = TBy(x) \Rightarrow ATy(x) = TBy(x), \forall y(x) \in D(B), T^{-1}AT = B,$$

it was required to prove.

Lemma 2.2. If $H=L_2(0,1)$, and

(1) $q(x) + q(1-x) = 0$;
(2) $Sy(x) = y(1-x), \forall y(x) \in L_2(0,1)$,

then the operators SA and SB are similar to each other, where

$$A = \frac{d}{dx}, D(A) = \{y(x) \in W_2^1, y(0) = 0\}$$
$$B = \frac{d}{dx} + q(x), D(B) = \{z(x) \in W_2^1, z(0) = 0\}$$

Proof. Let

$$Ty(x) = e^{\int_0^x q(t)dt} \cdot y(x),$$

then, by Lemma 1, we have $AT = TB, \forall y(x) \in D(B)$. Acting using the operator S on this equality we obtain

$$SAT = STB.$$

To prove the lemma it suffices that the operators S and T commute. Let us verify that when the condition, $q(x) + q(1-x) = 0$, is satisfied, the operators S and T commute.

$$STy(x) = Se^{\int_0^x q(t)dt} \cdot y(x) = e^{\int_0^x q(t)dt} \cdot y(1-x),$$
$$TSy(x) = Ty(1-x) = e^{\int_0^x q(t)dt} \cdot y(1-x),$$

If $STy(x) = TSy(x)$,то $e^{\int_0^x q(t)dt} = e^{\int_0^{1-x} q(t)dt}$, $\int_0^x q(t)dt - \int_0^{1-x} q(t)dt = 0$.

Differentiating the last equation, we obtain

$$q(x) + q(1-x) = 0.$$

The previous equality follows from the last equality. In fact, if

$$q(x)+q(1-x)=0,$$

$$\int_0^x q(t)dt-\int_0^x q(1-t)dt=0, \tag{7}$$

$$\int_0^x q(1-t)dt=\begin{vmatrix} t=1-\xi \\ dt=-d\xi \end{vmatrix}=-\int_0^{1-x} q(\xi)d\xi=\int_{1-x}^1 q(\xi)d\xi=\int_0^1 q(t)dt-\int_0^{1-x} q(t)dt; \tag{8}$$

$$\int_0^1 q(t)dt=\begin{vmatrix} t=1-\xi \\ dt=-d\xi \end{vmatrix}=-\int_1^0 q(1-\xi)d\xi=\int_0^1 q(1-\xi)d\xi=|q(1-\xi)=-q(\xi)|=$$

$$=\int_0^1 q(\xi)d\xi=-\int_0^1 q(t)dt,\Rightarrow \int_0^1 q(t)dt=0. \tag{9}$$

It is obvious that (9), (8), and (7) imply the equality

$$\int_0^x q(t)dt-\int_0^{1-x} q(t)dt=0.$$

From the last equality implies the equality $ST=TS$. The lemma is proved.

Lemma 2.3. Let

$$H=L_2(0,1),$$

$$q(x)+q(1-x)=0,$$

and

$$A=\frac{d}{dx}, D(A)=\{y(x)\in W_2^1, y(0)=0\}$$
$$B=\frac{d}{dx}+q(x), D(B)=\{z(x)\in W_2^1, z(0)=0\}$$

Then the spectra of the operators *SA* and *SB* coincide.

Proof. By Lemma 2, we have the equality

$$SAT=STB,\Rightarrow SA=TSBT^{-1},$$

where

$$Ty(x)=e^{\int_0^x q(t)dt}\cdot y(x).$$

Then

$$SA-\lambda I=T(SB-\lambda I)^{-1}.\Rightarrow (SA-\lambda I)^{-1}=T(SB-\lambda I)^{-1}T^{-1}.$$

Consequently, the resolvent sets of the operators *SA* and *SB* coincide, so their spectra also coincide.

Now we investigate the spectrum of the operator SA, in view of their importance for applications, we give detailed calculations.

Lemma 2.4. If

H=$L_2(0,1)$, $Sy(x)=y(1-x)$,

$$A = \frac{d}{dx}, D(A) = \{y(x) \in W_2^1, y(0) = 0\},$$

that the operator *SA* has an infinite set of eigenvalues

$$\lambda_n = (-1)^n \left(n\pi + \frac{\pi}{2}\right), n = 0, \pm 1, \pm 2, ...,$$

and their corresponding eigenfunctions

$$u_n = \sqrt{2} sin\left(n\pi + \frac{\pi}{2}\right) x, B_n = const,$$

which form an orthonormal basis of the space $H=L_2(0,1)$.
Proof. Let then $Au = \mu Su$, therefore, we are dealing with a generalized spectral problem:

$$\begin{cases} u'(x) = \mu \cdot u(1-x), \\ u(0) = 0. \end{cases} \quad (10)$$

Differentiating equations (10), we obtain

$$u''(x) = -\mu \cdot u'(1-x) = \mu \cdot \mu \cdot u(x) = -\mu^2 \cdot u(x), \Rightarrow$$

$$\begin{cases} u''(x) = \mu^2 \cdot u(x), & (11) \\ u(0) = 0, u'(1) = 0. & (12) \end{cases}$$

The general solution of equation (11) has the form

$$u(x) = Acos\mu x + Bsin\mu x, A, B - const \quad (13)$$

Substituting (13) into the boundary conditions (12), we obtain

$$u(0) = A = 0, u'(1) = [-\mu Asin\mu x + \mu Bcos\mu x]_{x=1} = \mu \cdot Bcos\mu = 0.$$

Since, B≠0, then the eigenvalues of problem (11) + (12) are found from equation

$$\Delta(\mu) = \mu cos\mu x = 0, \quad (14)$$

The trivial solution $u(x) \equiv 0$ corresponds to the value $\mu=0$, therefore it is not an eigenvalue. From the equation, $cos\mu=0$, we find the eigenvalues of problem (11) + (12).

$$\mu_n = n\pi + \frac{\pi}{2}, n = 0, \pm 1, \pm 2, ... \quad (15)$$

The square of each eigenvalue of problem (10) is an eigenvalue of the Sturm-Liouville problem (11) - (12), and the corresponding eigenfunctions coincide. But problem (11) - (12) can have other eigenvalues and corresponding eigenfunctions, so it is expedient to directly verify the eigenfunctions obtained. Substituting the eigenfunctions, $u_n(x)=B_n sin\mu_n x$, B_n-*const* into equation (10), we have

$$u_n'(x) = \mu_n \mathrm{B}_n cos\mu_n x,$$

$$u_n(1-x) = \mathrm{B}_n \cdot sin\mu_n(1-x) = \mathrm{B}_n \cdot sin(\mu_n - \mu_n x) =$$

$$= \mathrm{B}_n \cdot sin\mu_n \cdot cos\mu_n x - cos\mu_n \cdot sin\mu_n x = \mathrm{B}_n \cdot sin\left(n\pi + \frac{\pi}{2}\right) x \cdot cos\mu_n x = \mathrm{B}_n \cdot cosn\pi \cdot cos\mu_n x =$$
$$= (-1)^n \mathrm{B}_n \cdot cos\mu_n x.$$

Consequently,

$$u_n'(x) = (-1)^n \mu_n u_n(1-x),$$

Where

$$\mu_n = n\pi + \frac{\pi}{2}, n = 0, \pm 1, \pm 2, \ldots.$$

Let us show the completeness of the system of eigenfunctions obtained. Suppose that for some, $f(x) \in L_2(0,1)$, the equalities

$$\int_0^1 f(x) u_n(x) dx = 0, n = 1,2, \ldots .$$

are exist. Then

$$\int_0^1 f(x) \sin\left(n\pi + \frac{\pi}{2}\right) x dx = 0, n = 0, \pm 1, \pm 2, \ldots,$$
$$\int_0^1 f(x) \sin\left(-n\pi + \frac{\pi}{2}\right) x dx = 0, n = 0, \pm 1, \pm 2, \ldots.$$

Adding these two equalities, we have

$$\int_0^1 f(x) cos\frac{\pi x}{2} \sin\pi x dx = 0, n = 0, \pm 1, \pm 2, \ldots.$$

Hence, in view of the completeness of the system of functions $\{\sin n\pi x\}$ in the space $L_2(0,1)$, we obtain $f(x)cos\frac{\pi x}{2} = 0$ almost everywhere, hence $f(x) = 0$ almost everywhere.

The orthogonality of the resulting system is verified by direct calculation

$$\int_0^1 \sin\left(n\pi + \frac{\pi}{2}\right) x \cdot \sin\left(m\pi + \frac{\pi}{2}\right) x dx =$$
$$= \frac{1}{2} \int_0^1 [\cos(n-m)\pi x - \cos(n+m+1)\pi x] dx =$$

$$= \frac{1}{2} \left[\frac{\sin(n-m)\pi x}{(n-m)\pi} - \frac{\sin(n+m+1)\pi x}{(n+m+1)\pi} \right] \Bigg|_0^1 = 0, \text{при } n \neq m.$$

Calculating the norm of the eigenfunctions, we have

$$\|u_n\|^2 = 2 \cdot \int_0^1 sin^2\left(n\pi + \frac{\pi}{2}\right) x dx = \int_0^1 [1 - cos(2n\pi + \pi)x] dx = 1.$$

Lemma 2.4 is proved.

Now we are able to prove the following theorem.

Theorem 2.1. If $H = L_2(0,1)$ and$q(x)$ is a continuous real function satisfying the condition,

$$q(x) + q(1-x) = 0,$$

then the eigenfunctions of the boundary value problem

$$\begin{cases} y'(x) + q(x)y(x) = \lambda y(1-x), \\ y(0) = 0 \end{cases} \tag{14}$$

form a Riesz basis in $L_2(0,1)$.

Proof. Let $u_n(x)$ be the eigenfunctions of the boundary-value problem (10), then the functions

$$y_n(x) = e^{-\int_0^x q(t)dt} \cdot u_n(x)$$

will be the eigenfunctions of problem (14). In fact,

$$y_n'(x) = u_n'(x)e^{-\int_0^x q(t)dt} - q(x)u_n(x)e^{-\int_0^x q(t)dt}, \Rightarrow$$

$$y_n'(x) + y_n(x)q(x) = u_n'(x)e^{-\int_0^x q(t)dt}.$$

Operating the operator $Sy(x) = y(1-x)$ by this equality, and taking into account the conditions of the theorem, we have

$$s[y_n'(x) + y_n(x)q(y)] = u_n'(x)e^{-\int_0^x q(t)dt} = \mu_n u_n(x)e^{-\int_0^x q(t)dt} = \mu_n y_n(x), \Rightarrow$$

$$y_n'(x) + y_n(x)q(y) = \mu_n y_n(1-x), y_n(0) = 0.$$

It remains only to note that the operator

$$Tu_n(x) = e^{-\int_0^x q(t)dt} \cdot u_n(x)$$

linear bounded, and invertible operator in the space $L_2(0,1)$. The theorem is proved.

We now proceed to the solution of the problem posed earlier, for this we first solve the spectral problem

$$\begin{cases} \left[\frac{\partial}{\partial x} + p(x)\right]\left[\frac{\partial}{\partial y} + q(y)\right]u(x,y) = \lambda u(1-x, 1-y) & (15) \\ u|_{x=0} = 0, u|_{y=1} = 0. & (16) \end{cases}$$

We seek solutions of this problem in the form

$$u(x,y) = v(x) \cdot \omega(y). \tag{17}$$

Then from the boundary condition we have

$$u|_{x=0} = v(0) \cdot \omega(y) = 0, \Rightarrow v(0) = 0;$$
$$u|_{y=1} = v(x) \cdot \omega(1) = 0, \Rightarrow \omega(1) = 0.$$

Substituting (17) into equation (15), we have

$$\frac{\left[\frac{\partial}{\partial x}+p(x)\right]v(x)}{v(1-x)}\cdot\frac{\left[\frac{\partial}{\partial y}+q(y)\right]}{\omega(1-y)}=\lambda.$$

Dividing the variables, we obtain two spectral problems:

I. $\begin{cases} v'(x)+p(x)v(x)=\mu v(1-x), \\ v(0)=0. \end{cases}$

II. $\begin{cases} \omega'(x)+q(y)\omega(y)=\nu\omega(1-y), \\ \omega(1)=0. \end{cases}$

If $u(y)$ is a solution of the spectral problem

$\begin{cases} u'(y)=\nu u(1-y), \\ u(1)=0. \end{cases}$ (18)

и $q(y)+q(1-y)=0$, then the function

$$\omega(y)=e^{\int_y^1 q(\xi)d\xi}u(y)$$

is a solution of the spectral problem II. Indeed

$$\omega'(y)=u'(y)e^{\int_y^1 q(\xi)d\xi}-q(y)e^{\int_y^1 q(\xi)d\xi}u(y),\Rightarrow$$

$$\omega'(y)+q(y)\omega(y)=u'(y)e^{\int_y^1 q(\xi)d\xi}.$$

Let $Sy=y(1-x)$, then

$$S\left[\omega'(y)+q(y)\omega(y)\right]=e^{\int_y^1 q(\xi)d\xi}u'(1-y)=e^{\int_y^1 q(\xi)d\xi}\cdot\lambda u(1-y)=\lambda\omega(y),$$

$$\omega'(y)+q(y)\omega(y)=\lambda\omega(1-y).$$

Further, $u(1)=0$ implies $\omega(1)=0$, so that it remains for us to solve the problem (18)

$$\begin{cases} u'(x)=\nu u(1-x), \\ u(1)=0. \end{cases}$$

To use the results already known, we make a change of variable, assuming

$$v(x)=u(1-x), v'(x)=-u'(1-x), -v'(1-x)=u',$$

$$\begin{cases} -v'(1-x)=\nu v(x) \\ v(0)=0 \end{cases}\Rightarrow\begin{cases} v'(1-x)=-\nu v(x), \\ v(0)=0. \end{cases}$$

From Lemma 4 we know that

$v_n(x)=\mathrm{B}_n\cdot sin\left(n\pi+\frac{\pi}{2}\right)x$,therefore,

$$v_n(x) = v_n(1-x) = \mathrm{B}_n \cdot sin\left(n\pi + \tfrac{\pi}{2}\right)(1-x) = \mathrm{B}_n \cdot sin\left[n\pi + \tfrac{\pi}{2} - \left(n\pi + \tfrac{\pi}{2}\right)x\right] =$$

$$= \mathrm{B}_n \cdot cos\left(n\pi + \tfrac{\pi}{2}\right)x \cdot cosn\pi = (-1)^n \mathrm{B}_n \cdot cos\left(n\pi + \tfrac{\pi}{2}\right)x.$$

We compute the eigenvalues

$$v_n' = \mathrm{B}_n\left(n\pi + \frac{\pi}{2}\right) \cdot cos\left(n\pi + \frac{\pi}{2}\right)x,$$

$$v_n(1-x) = \mathrm{B}_n \cdot sin\left(n\pi + \frac{\pi}{2}\right)(1-x) = (-1)^n \mathrm{B}_n cos\left(n\pi + \frac{\pi}{2}\right)x,$$

$$v_n' = \left(n\pi + \frac{\pi}{2}\right)(-1)^n v_n(1-x) = -\left(n\pi + \frac{\pi}{2}\right)(-1)^{n+1} v_n(1-x), \Rightarrow$$

$$v_n = (-1)^{n+1}\left(n\pi + \frac{\pi}{2}\right).$$

Thus, the solution of the spectral problem (18) is the function

$$u_n(y) = c_n cos\left(n\pi + \frac{\pi}{2}\right)y,$$

but by the eigenvalues of the number: $v_n = (-1)^{n+1}\left(n\pi + \frac{\pi}{2}\right)$, where c_n- are the normalization coefficients. Let us calculate these coefficients

$$\|u_n\|^2 = |c_n|^2 \int_0^1 cos^2\left(n\pi + \tfrac{\pi}{2}\right)ydy = \tfrac{|c_n|^2}{2}\int_0^1 [1 + cos(2n\pi + \pi)y]dy =$$

$$= \frac{|c_n|^2}{2}y + \frac{sin(2n\pi + \pi)y}{2n\pi + \pi}\Bigg|_0^1 \frac{|c_n|^2}{2} = 1, c_n = \sqrt{2}.$$

For complete certainty, we verify the orthogonality of these eigenfunctions

$$(u_n, u_m) = 2\int_0^1 cos\left(n\pi + \frac{\pi}{2}\right)y \cdot cos\left(m\pi + \frac{\pi}{2}\right)ydy =$$

$$= \int_0^1 \left[cos\left(n\pi + \frac{\pi}{2} + m\pi + \frac{\pi}{2}\right)y + cos\left(n\pi + \frac{\pi}{2} - m\pi - \frac{\pi}{2}\right)y\right]dy =$$

$$= \int_0^1 \{cos[(n+m)\pi + \pi]y + cos(n-m)\pi y\}dy =$$

$$= \left[\tfrac{sin(n+m+1)y}{n+m+1} + \tfrac{sin(n-m)\pi y}{n-m}\right]\Big|_0^1 = 0, \text{при } n \neq m.$$

We have proved the following Lemma 2.5.
Lemma 2.5. The eigenfunctions of the spectral problem

$$\begin{cases} \omega'(y) + q(y)\omega(y) = \nu\omega(1-y), y \in (0,1) \\ \omega(1) = 0, \end{cases}$$

$$q(y) + q(1-y) = 0$$

is a function

$$\omega(y) = 2e^{\int_y^1 q(\xi)d\xi} \cdot cos\left(n\pi + \frac{\pi}{2}\right)y, n = 0, \pm 1, \pm 2, \ldots$$

and eigenvalues are

$$\upsilon_n = (-1)^{n+1} \cdot \left(n\pi + \frac{\pi}{2}\right), n = 0, \pm 1, \pm 2, \ldots$$

Theorem 2.2. If $H = L_2(0,1)$ and $q(x)$-continuous real function, satisflying condition.

$$q(x) + q(1-x) = 0,$$

then the eigenfunctions of a boundary value problems

$$\begin{cases} y'(x) + q(x)y = \nu y(1-x), \\ y(1) = 0 \end{cases}$$

form a Riesz basis of the space $L_2(0,1)$.

The proof of the theorem follows in an obvious way from Lemma 2.5. We summarize the results of the lemmas obtained [2.1-2.5], in the form of the following theorem,

Theorem 2.3. If

$$H = L_2(0,1) \text{ и}$$

$$p(x) + p(1-x) = 0,$$

$$q(y) + q(1-y) = 0$$

then the spectral problem

$$\begin{cases} \left[\frac{\partial}{\partial x} + p(x)\right]\left[\frac{\partial}{\partial y} + q(y)\right]u(x,y) = \lambda u(1-x, 1-y) \\ u|_{x=0} = 0, u|_{y=1} = 0 \end{cases}$$

has an infinite set of eigenvalues:

$$\lambda_{nm} = \pi^2(-1)^{n+m+1}\left(n + \frac{1}{2}\right)\left(m + \frac{1}{2}\right), n, m = 0, \pm 1, \pm 2, \ldots$$

and the corresponding eigenfunctions:

$$u_{nm}(x,y) = 2e^{\int_y^1 q(t)dt - \int_0^x p(t)dt} \cdot sin\left(n\pi + \frac{\pi}{2}\right)x \cdot cos\left(m\pi + \frac{\pi}{2}\right)y,$$

which form a Riesz basis of $L_2(D)$.

3. Results of the study Now we return to the original problem (3) + (4). Working Operator $S: Su(x,y) = u(1-x, 1-y)$ on equation (3), we obtain

$$S\left[\frac{\partial}{\partial x} + p(x)\right]\left[\frac{\partial}{\partial y} + q(y)\right]u(x,y) = Sf(x,y) = f(1-x, 1-y). \quad (19)$$

Expanding the function $u(x,y)$, and $f(1-x,1-y)$ in the eigenfunctions of the spectral problem (15) + (16), we have

$$f(1-x,1-y)=\sum_{n,m=-\infty}^{+\infty} f_{nm}u_{nm}(x,y),$$

$$u(x,y)=\sum_{n,m=-\infty}^{+\infty} a_{nm}\,u_{nm}(x,y).\ (20)$$

where f_{nm}, a_{nm}- are the corresponding Fourier coefficients. Substituting (20) into (19), we obtain

$$\sum_{n,m=-\infty}^{+\infty} \lambda_{nm}a_{nm}\,u_{nm}(x,y)=\sum_{n,m=-\infty}^{+\infty} f_{nm}\,u_{nm}(x,y),\Rightarrow$$

$$a_{nm}=\frac{f_{nm}}{\lambda_{nm}}.$$

Consequently,

$$u(x,y)=\sum_{n,m=-\infty}^{+\infty}\frac{f_{nm}}{\lambda_{nm}}u_{nm}(x,y).$$

Theorem 3.1. If $H=L_2(0,1)$ and

$$(a)\,p(x)+p(1-x)=0,(b)q(y)+q(1-x)=0,$$

then the Goursat problem

$$\begin{cases}\left[\frac{\partial}{\partial x}+p(x)\right]\left[\frac{\partial}{\partial y}+q(y)\right]u(x,y)=f(x,y),(x,y)\in D\\ u|_{x=0}=0,u|_{y=1}=0\end{cases}$$

is strongly solvable in the space $L_{2,\rho}(D)$with weight, and for the solution $u(x,y)$we have the representation

$$u(x,y)=\sum_{n,m=-\infty}^{+\infty}\frac{f_{nm}}{\lambda_{nm}}u_{nm}(x,y),$$

$$u_{nm}(x,y)=2exp\left[\int_y^1 q(t)dt-\int_0^x p(t)dt\right]\cdot sin\left(n\pi+\frac{\pi}{2}\right)x\cdot cos\left(m\pi+\frac{\pi}{2}\right)y,$$

$$\lambda_{nm}=\pi^2(-1)^{n+m+1}\left(n+\frac{1}{2}\right)\left(m+\frac{1}{2}\right),n,m=0,\pm1,\pm2,\dots$$

where f_{nm}-are the Fourier coefficients of the function $f(1-x,1-y)$ in the system $\{u_{nm}\}$.The scalar product in the space $L_{2,\rho}(D)$has the form

$$(f,g)=\int_0^1\int_0^1 exp\left[\int_y^1 q(t)dt-\int_0^x p(t)dt\right]f(x,y)g(x,y)dxdy.$$

Many people know the following lemma; nevertheless, for the sake of completeness, we give its proof.

Lemma 3.1.
Let $q(x)$be a periodic function with period 1, that is, $q(x+1)=q(x)$.Then in order for the function

$$Q(x)=\int_0^x q(t)dt$$

was a periodic function with a period equal to 1-c is necessary and sufficient that

$$\int_0^1 q(t)dt=0.$$

Proof.
(a) Necessity. Let Q (x) be a periodic function withperiod equal to one;
Q (x) = Q (1 + x). Then

$$\int_0^x q(t)dt=\int_0^{1+x} q(t)dt=\int_0^1 q(t)dt+\int_1^{1+x} q(t)dt;$$

$$\int_1^{1+x} q(t)dt=\begin{vmatrix} t=1+\xi \\ dt=d\xi \end{vmatrix}=\int_0^x q(1+\xi)d\xi=\int_0^x q(\xi)d\xi.$$

Следовательно

$$\int_0^x q(t)dt=\int_0^1 q(t)dt+\int_0^x q(\xi)d\xi.$$

Hence it is obvious that$\int_0^1 q(t)dt=0$.
(b) The sufficiency. Assume that the following equalities hold: $q(x)=q(x+1)$, $\int_0^1 q(t)dt=0$.
Then

$$Q(1+x)=\int_0^{1+x} q(t)dt=\int_0^1 q(t)dt+\int_1^{1+x} q(t)dt=$$

$$=\int_1^{1+x} q(t)dt=\begin{vmatrix} t=1+\xi \\ dt=d\xi \end{vmatrix}=\int_0^x q(1+\xi)d\xi=\int_0^x q(\xi)d\xi=Q(x).$$

and it was required to prove.

Corollary 3.1. If p (x) and q (y) are real continuous functions satisfying the following conditions:

$$p(x)+p(1-x)=0, q(y)+q(1-y)=0,$$

functions

$$P(x)=\int_0^x p(t)dt, Q(y)=\int_y^1 q(t)dt$$

are periodic with periods equal to one.
Proof. Lets show that if condition (1) is satisfied, we have

$$\int_0^1 p(t)dt = 0.$$

From condition (1), we have

$$\int_0^1 p(t)dt + \int_0^1 p(1-t)dt = 0; \int_0^1 p(1-t)dt = \begin{vmatrix} \xi = 1-t \\ d\xi = -dt \\ t = 1-\xi \end{vmatrix} = -\int_1^0 p(\xi)d\xi = \int_0^1 p(\xi)d\xi,$$

Consequently,

$$2\int_0^1 p(t)dt = 0 \Rightarrow \int_0^1 p(t)dt = 0\,.$$

Further,

$$Q(y) = \int_0^1 q(t)dt + \int_0^y q(t)dt = -\int_0^y q(t)dt.$$

Corollary 3.2. The eigenfunctions of the spectral problem (15) + (16) are periodic with period T = 2.
Proof.

$$u_{nm}(x,y) = 2exp\left[\int_y^1 q(t)dt - \int_0^x p(t)dt\right] \cdot sin\left(n\pi + \frac{\pi}{2}\right)x \cdot cos\left(m\pi + \frac{\pi}{2}\right)y,$$

$$u_{nm}(x+2,y+2) == 2exp\left[\int_y^1 q(t)dt - \int_0^x p(t)dt\right] \cdot sin\left[2n\pi + \pi + \left(n\pi + \frac{\pi}{2}\right)x\right] \cdot$$
$$cos\left[2m\pi + \pi + \left(m\pi + \frac{\pi}{2}\right)y\right] =$$
$$= 2exp\left[\int_y^1 q(t)dt - \int_0^x p(t)dt\right] \cdot sin\left(n\pi + \frac{\pi}{2}\right)x \cdot cos\left(m\pi + \frac{\pi}{2}\right)y = u_{nm}(x,y).$$

We now state the resulting final theorem.

Theorem 3.2. Let, $H = L_2(D)$,and $p(x), q(y), f(x,y)$be real continuous functions. If the following conditions are true:

(a) $p(x) + p(1-x) = 0,$
(b) $q(y) + q(1-x) = 0,$
(c) $f(x,y) \in C_0(D);$

then the Goursat problem

$$\left[\frac{\partial}{\partial x} + p(x)\right]\left[\frac{\partial}{\partial y} + q(y)\right]u(x,y) = f(x,y), (x,y) \in D$$
$$u|_{x=0} = 0, u|_{y=1} = 0$$

has a unique periodic solution, with a period T = 2.

4. Discussion. The incorrectness of the Dirichlet problem of the wave equation $u_{xx} - u_{yy} = 0$in region D [see Pic.2] is well known, from the operator's point of view the wave operator has a continuous spectrum, that is, zero is an infinite eigen value, the periodic problem has an analogous property, so we have a periodic problem turned their attention to Goursat's problem.

5. Conclusions. Wave equations describe wave processes: propagation of sound, electromagnetic waves, waves on water, radio waves, etc. There are cases when waves of small amplitude form giant waves. This phenomenon is due to the duration of the wave propagation process, therefore the problem of stabilizing the solutions of the wave equation as $t \to +\infty$is of great practical importance. One of the signs of wave stabilization is their periodicity. We have established that if the external perturbation is localized, i.e. is finite, and the coefficients of the wave equation are periodic and odd, then the solution of the Goursat problem admits a periodic extension to the whole plane of independent variables.

REFERENCES

[1] Hadamard J. On some topics connected with linear partial differential equations, Proc.Benares Math. Soc, **1921**, 3, p. 39-48.

[2] Burgin D, Duffin R. The Dirichlet problem for the vibrating string equation,Bull.Amer. Math. Soc, **1939**, iv. 45, p. 851-858.

[3] Арнольд В. И. Малые знаменатели, I. Изв. АН СССР. Сер. матем., **1961**, т. 25, № 1,с. 21-86.

[4] Бобик О.И.,Бондарчук П.И., Пташник Б. И. Элементы качественной теории дифференциальных уравнений в частных производных. Киев, **1972**. 175с.

[5] SleemanB. D. Boundary value problems for the vibrating string equation.- Proc. Roy.Soc. Edinburgh, **1981**, v. A-88, Us. 1-2, p. 185-194.

[6] Мосолов П. П. О задаче Дирихле для уравнений в частных производных. Известия высших учебных заведений,**1960** Математика № 3 (16), с. 213-218.

[7] Соболев С.Л.Пример корректной краевой задачи для уравнения колебания струны с данными на всей границе,ДАН-СССР, **1956**, т. 109, JM» 4, с. 707-709.

[8] Березанский Ю. М. Разложение по собственным функциям самосопряженных операторов.Киев, **1965**. 800 с.

[9] Александрян Р. А. Спектральные свойства операторов, порожденных системами дифференциальных уравнений типа С Л .Соболева. -Тр. Моск. матем. о-ва, **1960**, т. 9, с. 455-505.

[10] Кальменов Т.Ш. О регулярных краевых задачах для волнового уравнения, Диффенц. уравнения. **1981**, т. 17, №6, с. 1105-1121.

[11] Кальменов Т.Ш.О спектре одной самосопряженной задачи для волнового уравнения, Весник А.Н Каз ССР, **1982**, №2, с. 63-66.

[12] Садыбеков М.И, Кальменов Т.Ш. О задаче Дирихле и нелокальных краевых задачах для волнового уравнения, Диффенц. уравнения, **1990**, т. 26, №1, с. 60-65.

[13] Кальменов Т. Ш. Краевые задачи для линейных уравнений в частных производных гиперболического типа , Шымкент: Ғылым, **1993**, 327 с.

[14] Бурский В. П. Краевые задачи для гиперболического уравнения второго порядка в круге, Известия. вузов. Матем., **1987**, номер 2,22–29.

[15] Бурский В.П.О ядре дифференциального оператора с постоянными коэффициентами младшего порядка в круге, ВИНИТИ, № 3792-82 Деп., **1982**.

[16] Саргсян Г.А.О полиномиальных решениях задачи Дирихле для гиперболических уравнений с постоянными коэффициентами в круге, Док лады национальной академии наук Армении, Том 112, **2012**, № 4.

[17] Кальменов Т.Ш., Ахметова С.Т, Шалданбаев А.Ш. К спектральной теории уравнений с отклоняющимся аргументом // Математический журнал, Алматы. **2004**. Т. 4, № 3. С. 41-48.

[18] Ибраймкулов А.М. О спектральных свойствах краевой задачи для уравнения с отклоняющимся аргументом // Известия АН.Каз.ССР, сер.физ.-мат. **1988**. № 3. С. 22-25.

[19] Kal'menov T. Sh., and Shaldanbaev A. Sh., On a criterion of solvability of the inverse problem of heat conduction, Journal of Inverse and Ill-Posed Problems 18, 352-369 (**2010**).

[20] Orazov I, A. Shaldanbayev, and M. Shomanbayeva, About the Nature of the Spectrum of the Periodic Problem for the Heat Equation with a Deviating Argument, Abstract and Applied Analysis, Volume 2013 (**2013**), Article ID 128363, 6 pages, http://dx.doi.org/10.1155/2013/128363.

[21] Шалданбаев А.Ш. Спектральные разложения корректных-некорректных начально краевых задач для некоторых классов дифференциальных уравнений. Монография. 193с,LAP LAMBEPT Academic Pyblishing. http://dnb.d-nb.de. Email:info@lap-publishing.com,Saarbrucken **2011** ,Germanu.

[22] Allaberen Ashyralyev, Abdizhahan M. Sarsenbi, Well-posedness of an Elliptic Equation with Involution, Electronic Journal of Di_erential Equations, Vol. **2015** (2015), No. 284, pp. 1{8.ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.eduftp ejde.math.txstate.edu.

[23] Садыбеков М.А.,Сарсенби А.М. Решение основных спектральных вопросов всех краевых задач для одного дифференциального уравнения первого порядка с отклоняющимся аргументом,Uzbek Mathematical Journal, **2007**, №3, pp.1-6.

[24] Sadybekov, M. A.; Sarsenbi, A. M.; Mixed problem for a differential equation with involution under boundary conditions of general form. AIP Conference Proceedings. Ed. Ashyralyev and A. Lukashov, A. Vol. 1470, 225-227, **2012**.

УДК 517.956.32

М.И. Ақылбаев[1], А. Бейсебаева[2], А. Ш. Шалданбаев[3]

[1]Региональный социально-гуманитарный университет, г. Шымкент;
[2]ЮКГУ им. М. Ауезова, г. Шымкент;
[3]ЮКГУим. М.Ауезова, г. Шымкент.

О ПЕРИОДИЧЕСКОМ РЕШЕНИИ ЗАДАЧИ ГУРСА ДЛЯ ВОЛНОВОГО УРАВНЕНИЯ СПЕЦИАЛЬНОГО ВИДА С ПЕРЕМЕННЫМИ КОЭФФИЦИЕНТАМИ

Аннотация. В данной работе решена задача Гурса в характеристическом четырехугольнике для волнового уравнения специального вида с переменными коэффициентами. Получено спектральное представление решения, не традиционное для таких вольтерровых задач. Для этого в качестве вввспомогоальной задачи использована спектральная задача для уравнения с отклонящимся аргументом. Показано, что опратор вида Su(x)=u(1-x) играет роль оператора Шмидта встречающиеся в разложениях вольтерровых операторов.

Ключевые слова: Вольтерровые операторы, индефинитная метрика, задача Гурса, операторы подобия, спектр, спектральное разложение, метод Фурье, ортогональный базис, теорема Гиьберта-Шмидта.

М.И.Ақылбаев[1], А. Бейсебаева[2], А. Ш. Шалданбаев[3],

[1]Аймақтық әлеуметтік-инновациялық университеті;
[2]ЮКГУ им. М.Ауезова, г. Шымкент;
[3]ЮКГУим. М.Ауезова, г. Шымкент

КОЭФФИЦИЕНТТЕРІ АЙНЫМАЛЫ ТҮРІ АРНАЙЫ ТОЛҚЫН ТЕҢДЕУІНІҢ ГУРСАЛЫҚ ЕСЕБІНІҢ ПЕРИОДТЫ ШЕШІМІ ТУРАЛЫ

Аннотация. Бұл еңбекте коэффициенттері айнымалы ал түрі арнайы толқын теңдеуіне қойылған Гурсаның есебі шешілді. Шешімнің спектрәлді кейпі табылды, мұндай жағдай вөлтерлі есептерге тән емес.Бұл үшін көмекші есеп ретінде аргументі ауытқыған дифференциалдық теңдеу қолданылды. Мынадай, Su(x)=u(1-x), операторлардың Шмидтің операторының қызметін атқаратыны көрсетілген

Тірек сөздер: Вөлтерлік операторлар, индефинитті метрика, Гурсаның есебі, ұқсастық опероторы, спектр, спектрәлдік таралым, Фүренің әдісі, ортогонәлді базис, Гілберт-Шмидтің теоремасы.

Information about authors:
Akylbaev MI - vice-rector Regional Social-Innovational University, Shymkent;
Beissebaeva A. - teacher of the department "Mathematics" M.Auezov South Kazakhstan State University, Shymkent;
Shaldanbayev A.Sh. - Doctor of physical and mathematical sciences, professor of the department "Mathematics" M.Auezov South Kazakhstan State University, Shymkent.