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NONLINEAR HOOKE LAW IN THE THEORY OF ELASTICITY OF INHOMOGENEOUS AND ANISOTROPIC BODIES

Abstract. Directly from the physical connection with the nonlinear Hooke law, the components of the stress tensor of a rigid deformable body and new nonlinear equations of the theory of elasticity with an asymmetric stress tensor are derived, as a special case we obtain equations with the linear Hooke's law. The Lame hypothesis and Lame's equations do not have a physical connection with Hooke's law, this is their falsehood. Lame took as a basis the approximate formula of the incomplete differential and suggested in his hypothesis the proportionality of the stress tensor components to the symmetrical half of the given incomplete differential of displacement, and the antisymmetric half of the differential is discarded, which is the result of the false symmetry of the Lame stress tensor. The new nonlinear equations are approximated by an explicit scheme, with the use of which the elastic state of a flat bar is numerically calculated with the normal and tangential stresses acting on the upper face. The same scheme is applied to the Lame equations. The obtained patterns of displacements distribution clearly demonstrate the difference in the solutions of the comparable systems of elasticity equations, as well as the discrepancy between the solution of the Lame equations for a given state of the deformed body. The falsity of Lame's equations is confirmed theoretically and physically.
Keywords: tensile, tangent, normal, stress, tensor.

1. Tangential stresses according to the generalized Hooke's law

Hooke's law is an assertion according to which the deformation arising in an elastic body, is proportional to the applied force. It was discovered in 1660 by the English scientist Robert Hooke.

It should be kept in mind that Hooke's law is satisfied only for small deformations. If the proportionality limit is exceeded, the relationship between stresses and strains becomes nonlinear. For many media, Hooke's law does not apply even for small deformations. The derivation of dynamic equations with an asymmetric stress tensor according to Hooke's linear law is given in [1]: $\mathbf{F} = k\mathbf{u}, k>0$, $\mathbf{F}_x = u\mathbf{i}, \mathbf{F}_y = v\mathbf{j}$, $\mathbf{F}_z = w\mathbf{k}$, $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ – vector of displacement, where $\mathbf{F}=\mathbf{F}_x+\mathbf{F}_y+\mathbf{F}_z$ – external force that causes displacement.

In inhomogeneous media composed of bodies with various elastic properties or in anisotropic bodies whose properties depend on direction, Hooke's law can be nonlinear

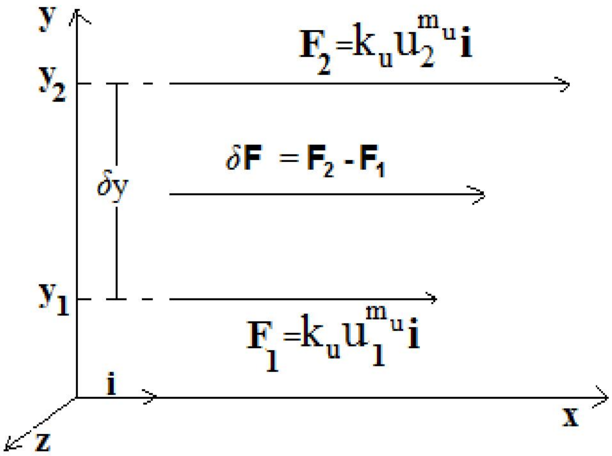
$$\mathbf{F} = k_u u^{m_u}\mathbf{i} + k_v v^{m_v}\mathbf{j} + k_w w^{m_w}\mathbf{k}, \ k_u > 0, k_v > 0, k_w > 0 \quad (1.1)$$

For exponents of 1 and $k_u = k_v = k_w = k$, (1.1) becomes Hooke's linear law for an isotropic medium, hence the exponents must be odd numbers [2], which will be confirmed below by the properties of hyperbolic equations of anguish. Suppose that on the plane y_1 force is associated with the movement under the generalized law $\mathbf{F}_1 = k_u u_1^{m_u}\mathbf{i}$, similarly force $\mathbf{F}_2 = k_u u_2^{m_u}\mathbf{i}$ on the surface $y_2 = y_1 + \delta y$, $\delta y > 0$.

The increments of forces and displacements between layers are equal:

$$\delta\mathbf{F} = \mathbf{F}_2 - \mathbf{F}_1 = k_u u_2^{m_u}\mathbf{i} - k_u u_1^{m_u}\mathbf{i} = k_u \delta u^{m_u}\mathbf{i},\ \delta u^{m_u} = u_2^{m_u} - u_1^{m_u} > 0.$$

Let $|\mathbf{F}_2| > |\mathbf{F}_1|$, in this case the force increment is directed along the x axis: $\delta\mathbf{F} \uparrow\uparrow \mathbf{i}$.



Linear density is introduced $\mathbf{f} = \delta\mathbf{F} / \delta y,\ \delta\mathbf{F} = \delta y \mathbf{f}$. By definition, the average tangent stress vector $\mathbf{p}_{yxav} = \dfrac{\delta\mathbf{F}}{\delta x \delta z}$ parallel and equally directed with a force that causes this voltage $\mathbf{p}_{yxav} \uparrow\uparrow \delta\mathbf{F}$, $\mathbf{p}_{yxav} \uparrow\uparrow \mathbf{f}$.

By the introduction of the proportionality coefficient, the following bonds are formed:

$$\mathbf{f} = k'\mathbf{p}_{yxav}, k'>0,\ \delta y\mathbf{f} = k'\mathbf{p}_{yxav}\delta y,\ \mathbf{p}_{yxav} \uparrow\uparrow \mathbf{i},$$

$$k'\mathbf{p}_{yxav}\delta y = k_u \delta u^{m_u}\mathbf{i}$$

This expression is multiplied scalar by the unit vector $\mathbf{i}$:

$$(k'\mathbf{p}_{yxav}\delta y, \mathbf{i}) = (k_u \delta u^{m_u}\mathbf{i}, \mathbf{i})$$

As result

$$(k'\mathbf{p}_{yxav}\delta y, \mathbf{i}) = k'|\mathbf{p}_{yxav}|\delta y|\mathbf{i}|\cos 0^o = k' p_{yxav}\delta y,\ (k_u \delta u^{m_u}\mathbf{i}, \mathbf{i}) = k_u \delta u^{m_u}$$

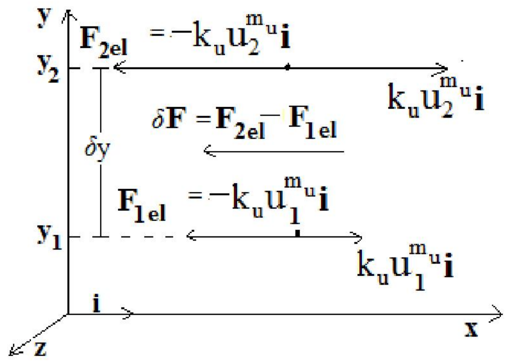
Equalities $k' p_{yxav}\delta y = k_u \delta u^{m_u}$, $p_{yxav} = \dfrac{k_u}{k'}\dfrac{\delta u^{m_u}}{\delta y}$ in the limit give a tangential stress

$$p_{yx} = \lim_{\delta y \to 0} \frac{k_u}{k'}\frac{\delta u^{m_u}}{\delta y} = \mu_u \frac{\partial u^{m_u}}{\partial y},\ \mu_u = \frac{k_u}{k'} > 0$$

Tangential stresses in other directions are obtained analogously:

$$p_{xy} = \mu_v \frac{\partial v^{m_v}}{\partial x}, p_{zx} = \mu_u \frac{\partial u^{m_u}}{\partial z}, p_{xz} = \mu_w \frac{\partial w^{m_w}}{\partial x}, p_{yz} = \mu_w \frac{\partial w^{m_w}}{\partial y}, p_{zy} = \mu_v \frac{\partial v^{m_v}}{\partial z}$$

The asymmetric shearing stress formulas are derived for causing stretching of the body of an external force $\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y + \mathbf{F}_z$.



The elastic force in the deformed body according to Newton's third law is equal to the external force with a minus sign $\mathbf{F}_{el} = -\mathbf{F}$. Consequently, the linear Hooke law for elastic forces will have the form $\mathbf{F}_{el} = -k\mathbf{u}, \mathbf{F}_{el} = -ku\mathbf{i} - kv\mathbf{j} - kw\mathbf{k}$. A similar representation for Hooke's nonlinear law

$$\mathbf{F}_{el} = -k_u u^{m_u}\mathbf{i} - k_v v^{m_v}\mathbf{j} - k_w w^{m_w}\mathbf{k}$$

Suppose that on the plane y1 the force is connected with the displacement along the nonlinear law $\mathbf{F}_{1el} = k_u u_1^{m_u}\mathbf{i}$, similarly acts force $\mathbf{F}_{2el} = k_u u_2^{m_u}\mathbf{i}$ on surface $y_2 = y_1 + \delta y,\ \delta y > 0$.

The increments of forces and displacements between layers are equal:

$$\delta\mathbf{F} = \mathbf{F}_{2el} - \mathbf{F}_{1el} = -k_u u_2^{m_u}\mathbf{i} + k_u u_1^{m_u}\mathbf{i} = -k_u \delta u^{m_u}\mathbf{i},$$
$$\delta u^{m_u} = u_2^{m_u} - u_1^{m_u} > 0.$$

Let $|\mathbf{F}_2| > |\mathbf{F}_1|$, in this case, the force increment is directed against the x axis: $\delta\mathbf{F} \uparrow\downarrow \mathbf{i}$. Linear density is introduced $\mathbf{f} = \delta\mathbf{F}/\delta y,\ \delta\mathbf{F} = \delta y\mathbf{f}$.

By definition, the average tangent stress vector $\mathbf{p}_{yxav} = \dfrac{\delta\mathbf{F}}{\delta\sigma}, \delta\sigma = \delta x\delta z$ parallel and equally directed with a force that causes this voltage $\mathbf{p}_{yxav} \uparrow\uparrow \delta\mathbf{F},\ \mathbf{p}_{yxav} \uparrow\uparrow \mathbf{f}$.

The input of the proportionality coefficient formed a bond:

$$\mathbf{f} = k'\mathbf{p}_{yxav}, k'>0,\ \delta y\mathbf{f} = k'\mathbf{p}_{yxav}\delta y,\ \mathbf{p}_{yxav} \uparrow\downarrow \mathbf{i},\ k'\mathbf{p}_{yxav}\delta y = -k_u\delta u^{m_u}\mathbf{i}$$

This expression is multiplied scalar by the unit vector $\mathbf{i}$:

$$(k'\mathbf{p}_{yxav}\delta y, \mathbf{i}) = -(k_u\delta u^{m_u}\mathbf{i}, \mathbf{i})$$

As a result

$$(k'\mathbf{p}_{yxav}\delta y, \mathbf{i}) = k'|\mathbf{p}_{yxav}|\delta y|\mathbf{i}|\cos 180^o = -k'p_{yxav}\delta y,\ -(k_u\delta u^{m_u}\mathbf{i}, \mathbf{i}) = -k_u\delta u^{m_u}$$

Equalities

$$k'p_{yxav}\delta y = k_u\delta u^{m_u},\ p_{yxav} = \frac{k_u}{k'}\frac{\delta u^{m_u}}{\delta y}$$

in the limit give a tangential stress

$$p_{yx} = \lim_{\delta y \to 0}\frac{k_u}{k'}\frac{\delta u^{m_u}}{\delta y} = \mu_u\frac{\partial u^{m_u}}{\partial y},\ \mu_u = \frac{k_u}{k'} > 0$$

Similarly, tangential stresses are obtained in other directions:

$$p_{xy}=\mu_v\frac{\partial v^{m_v}}{\partial x},p_{zx}=\mu_u\frac{\partial u^{m_u}}{\partial z},p_{xz}=\mu_w\frac{\partial w^{m_w}}{\partial x},p_{yz}=\mu_w\frac{\partial w^{m_w}}{\partial y},p_{zy}=\mu_v\frac{\partial v^{m_v}}{\partial z}$$

In this way, the stress formulas for external forces coincide with formulas of the action of elastic forces, therefore further conclusions are made only for external forces.

2. Relation of Normal Stresses to Hooke's Law

A similar argument establishes the formula of the component $\mathbf{p}^o_{xx}$ normal voltage $\mathbf{p}_{xx}=\lambda \mathrm{div}\mathbf{u}\ \mathbf{i}+\mathbf{p}^o_{xx}$.

Let the external forces be equal: $\mathbf{F}_1 = k_u u_1^{m_u}\mathbf{i}$ on surface x_1 and $\mathbf{F}_2=k_u u_2^{m_u}\mathbf{i}$ on surface $x_2=x_1+\delta x,\delta x>0.$

The increments of forces and displacements between layers are equal:

$$\delta\mathbf{F}=\mathbf{F}_2-\mathbf{F}_1=k_u u_2^{m_u}\mathbf{i}-k_u u_1^{m_u}\mathbf{i}=k_u\delta u^{m_u}\mathbf{i},\ \ \delta u^{m_u}=u_2^{m_u}-u_1^{m_u}>0.$$

By definition, the average stress vector $\mathbf{p}^o_{xx\,av}=\frac{\delta\mathbf{F}}{\delta\sigma},\delta\sigma=\delta y\delta z$ parallel and equally directed with a force that causes this voltage $\delta\mathbf{F}\uparrow\uparrow\mathbf{i}$, $|\mathbf{F}_2|>|\mathbf{F}_1|$.

Through the linear density $\mathbf{f}=\frac{\delta\mathbf{F}}{\delta x},\delta\mathbf{F}=\delta x\mathbf{f},\ \mathbf{f}=k''\mathbf{p}^0_{xx\,av}$ equalities $\delta\mathbf{F}=k''\delta x\mathbf{p}^0_{xx\,av}$, $k''\delta x\mathbf{p}^0_{xx\,av}=k_u\delta u^{m_u}\mathbf{i}.$

This expression is multiplied scalar by the unit vector $\mathbf{i}$:

$$(k''\delta x\mathbf{p}^0_{xx\,av},\mathbf{i})=(k_u\delta u^{m_u}\mathbf{i},\mathbf{i})$$

Vectors are parallel in structure $\mathbf{p}^0_{xx\,cp}\uparrow\uparrow\mathbf{i}$.

Therefore, they occur in scalar products

$$(k''\delta x\mathbf{p}^0_{xxav},\mathbf{i})=k''\delta x|\mathbf{p}^0_{xxav}|\cdot|\mathbf{i}|\cdot\cos 0^o=k''\delta x p^0_{xxav},\ (k_u\delta u^{m_u}\mathbf{i},\mathbf{i})=k_u\delta u^{m_u}.$$

The result is $k''\delta x p^0_{xxav}=k_u\delta u^{m_u}$, whence $p^0_{xxav}=\frac{k_u}{k''}\frac{\delta u^{m_u}}{\delta x}$.

In the limit, the formula for the component of the normal stress

$$p^0_{xx}=\lim_{\delta x\to 0}\frac{k_u}{k''}\frac{\delta u^{m_u}}{\delta x}=\mu_u\frac{\partial u^{m_u}}{\partial x},\ \ \mu_u=\frac{k_u}{k''}.$$

The same reasonings also show the components of normal stresses in other direc tions:

$$p^o_{ii}=\mu_i\frac{\partial u_i^{m_i}}{\partial x_i},\ \mu_i=\frac{k_i}{k''},i=1,2,3;\ u_1\equiv u,u_2\equiv v,u_3\equiv w,x_1\equiv x,x_2\equiv y,x_3\equiv z.$$

The same formulas for normal stresses are obtained for the elastic forces in a solid deformed body $\mathbf{F}_{упр} = -\mathbf{F}$.

Thus, the nonlinear Hooke law corresponds to an asymmetric stress tensor in a solid deformed body:

$$p_{ji}=\delta_{ji}\lambda \mathrm{div}\mathbf{u} + \mu_i\varepsilon_{ji}, \quad \varepsilon_{ji} = \frac{\partial u_i^{m_i}}{\partial x_j}, i,j=1,2,3, \tag{2.1}$$

$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ - the vector of displacement. In normal stresses $\lambda\delta_{ji}\mathrm{div}\mathbf{u}$ by Lama is preserved, δ_{ji} - the Kronecker symbol.

3. On falsifications and inapplicability of Lame's equations in the nonlinear theory of elasticity of anisotropic bodies

Linear equations of the theory of elasticity of a solid deformable body

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \rho_0 \mathbf{F} + (\lambda + \mu)\mathrm{graddiv}\mathbf{u} + \mu\Delta\mathbf{u} \tag{3.1}$$

constructed according to the hypothesis of Lame on the symmetry of the stress tensor

$$p_{ji} = \lambda\delta_{ji}\mathrm{div}\mathbf{u} + 2\mu\varepsilon_{ji}, \varepsilon_{ji} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), p_{ji} = p_{ij}, i,j=1,2,3, \tag{3.2}$$

λ, μ – the Lame coefficients.

The Lame hypothesis is that the elements of the strain tensor ε should be equal to the doubled symmetric strain rate tensor, that is, doubled the first half of the artificial formula

$$du_i = \sum_{j=1}^{3}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right]dx_j, i=1,2,3, \tag{3.3}$$

(the second antisymmetric half of (3.3) is ignored [3-11]).

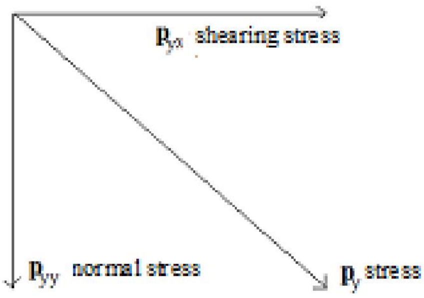
The formula (3.3) is formed from the incomplete differential of displacements

$$\tilde{d}u_i = \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j} dx_j, i=1,2,3, \tag{3.4}$$

(Here is the total differential: $d\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t}dt + \sum_{j=1}^{3}\frac{\partial \mathbf{u}}{\partial x_j}dx_j = \frac{\partial \mathbf{u}}{\partial t}dt + \tilde{d}\mathbf{u}$).

Thus, in [3-12] the Lame stress tensor (3.2) does not correspond and does not follow from Hooke's law. Obviously, already by construction, it is not suitable in anisotropic media, especially in the nonlinear Hooke's law.

The above physical method of the author of constructing the stress tensor of an elastic body according to Hooke's law is the opposite of the Lame hypothesis and exactly follows the definition given by Timoshchenko in [3]: "**The main task of the theory of elasticity is to find, according to the external forces acting on the solid body, those changes in shape that the body undergoes, and those internal elastic forces that, with these changes of form, arise between parts of the body."**



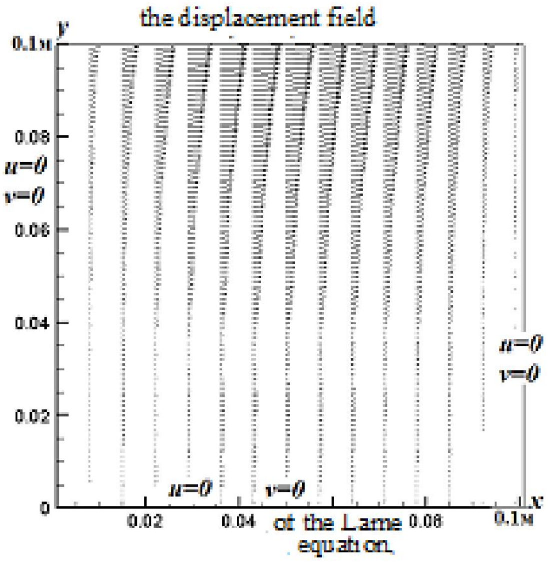


Figura 1

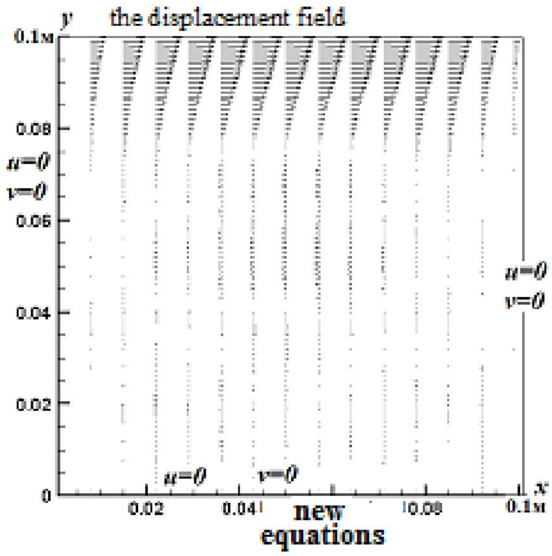


Figura 2

To compare Lame's equations (3.1) with new equations with asymmetric-a tensor of stresses in an isotropic body $\mu_{\rm u} = \mu_{\rm v} = \mu_{\rm w} = \mu$:

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \rho_0 \mathbf{F} + \lambda \mathrm{grad}\,\mathrm{div}\,\mathbf{u} + \mu \Delta \mathbf{u} \tag{3.5}$$

displacements in a square deformable bar measuring 0.1 m by 0.1 m have been calculated.

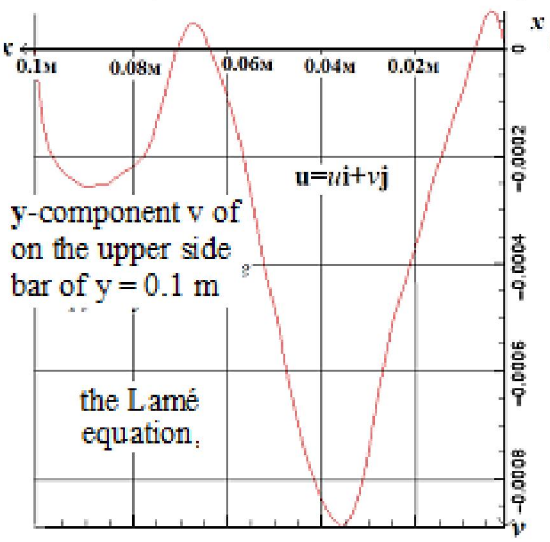


Figura 3

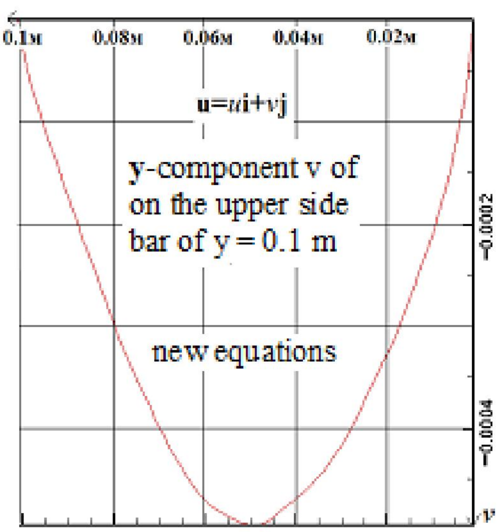


Figura 4

The vector of external force $\mathbf{p}_y = \mathbf{p}_{yx} + \mathbf{p}_{yy} = p_{yx}\mathbf{i} + p_{yy}\mathbf{j}$ is directed at an angle to the upper plane of the bar. In Fig. 1 and 2 there are the fields of displacement vectors $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$, in Fig. 3 and 4 transverse velocity diagrams on the upper side of the bar, all at time t = 121.38 s. Body density $\rho_0 = 7800$ кг/м^3. Specifically laid p_{yy}=-1Н/м^2, p_{yx}=10 Н/м^2. The other edges of the bar are rigidly fixed, the displacements on them are equal to zero. The Lame coefficients are chosen equal to $\lambda = 1$ кг/(с^{2}м), $\mu = 100$ kg/(с^{2}м). The two-dimensional Lame equations (3.1) and the new equations (3.5) are realized by explicit schemes [2] on a 100x100 grid with a time step equal to 0.0005s. A clear difference between the numerical solutions, especially the vertical displacements on the upper side of the bar in Fig. 3 and 4.

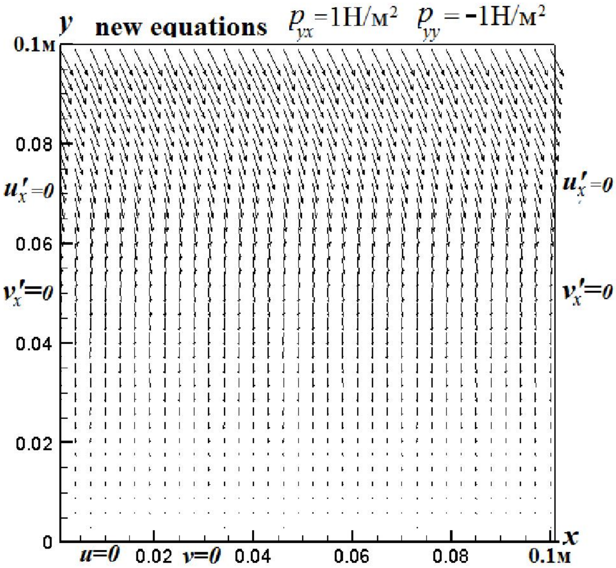


Figura 5

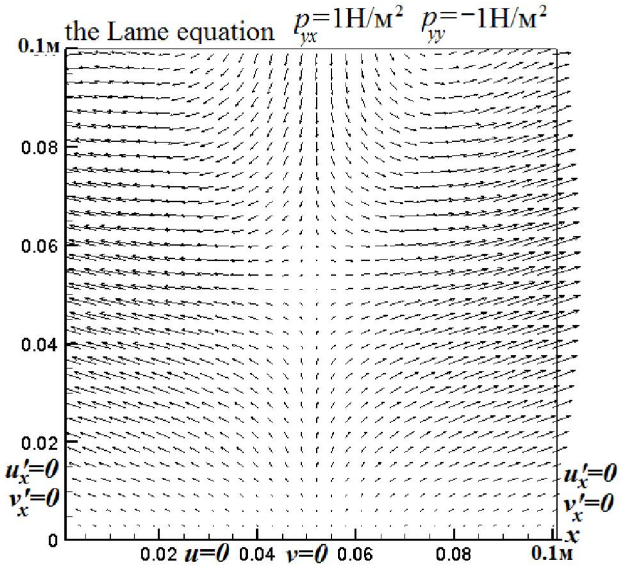


Figura 6

In Fig. 4, the movement of particles of the upper side of the bar y = 0.1 m occurs downward, which is confirmed by negative values of the transverse displacement v with respect to the new equations. In Fig. 3, Lame equations have positive values for the transverse component of the displacement, which contradicts the direction of the action of the external force.

In Fig. 5 the displacement field for the new equations; Fig. 6 the field of displacements by the Lamé equations. Stresses act on the entire upper side of the bar. The results of FIG. 1 ± 6 practically confirm the falsity of the Lame equations with a symmetric stress tensor.

4.Equations of nonlinear anisotropic theory of elasticity according to the generalized Hooke's law with an asymmetric stress tensor

Leaving from Hooke's linear law [1], the elements of the strain tensor $\varepsilon_{ji} = \dfrac{\partial u_i}{\partial x_j}$, i,j=1,2,3 are directly equal to the coefficients of the incomplete differential (3.4).

The asymmetric stress tensor of the generalized Hooke's law

$$p_{ji} = \delta_{ji}\lambda \operatorname{div}\mathbf{u} + \mu_i \varepsilon_{ji}, \quad \varepsilon_{ji} = \frac{\partial u_i^{m_i}}{\partial x_j}, i,j=1,2,3, \tag{4.1}$$

in the equations of the dynamics of a continuous medium in stresses

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \rho_0 F_i + \sum_{j=1}^{3} \frac{\partial p_{ji}}{\partial x_j}, i=1,2,3$$

form non-linear equations with exponents in accordance with the generalized Hooke's law

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \rho_0 F_i + \lambda \frac{\partial}{\partial x_i} \mathrm{div}\,\mathbf{u} + \mu_i \Delta u_i^{m_i}, i=1,2,3 \qquad (4.2)$$

5. Rationale for the oddness of entire exponents of degree

Equations (4.2) with Cartesian coordinates

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \rho_0 F_i + \lambda \frac{\partial}{\partial x_i} \mathrm{div}\,\mathbf{u} + \mu_i \sum_{j=1}^{3} \frac{\partial^2 u_i^{m_i}}{\partial x_j^2}, i=1,2,3$$

are represented in a differentiated form

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \rho_0 F_i + \lambda \frac{\partial}{\partial x_i} \mathrm{div}\,\mathbf{u} + \mu_i \frac{\partial}{\partial x_j} \left(\frac{\partial u_i^{m_i}}{\partial x_j}\right), i=1,2,3,$$

where $\frac{\partial u_i^{m_i}}{\partial x_j} = m_i u_i^{m_i-1} \frac{\partial u_i}{\partial x_j}$.

Obviously, the equivalent equations

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \rho_0 F_i + \lambda \frac{\partial}{\partial x_i} \mathrm{div}\,\mathbf{u} + \mu_i \frac{\partial}{\partial x_j} \left(m_i u_i^{m_i-1} \frac{\partial u_i}{\partial x_j}\right), i=1,2,3 \qquad (5.1)$$

are equations of hyperbolic type only for odd exponents of degree $m_i = 1;3;5;7;9$ и *т.д.*, $m_1 \equiv m_u, m_2 \equiv m_v, m_3 \equiv m_w$ for always $m_i u_i^{m_i-1} \ge 0$.

6. Explicit diagram of the equations of an anisotropic nonlinear theory of elasticity

The Cauchy-Dirichlet problem for the new equations

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \rho_0 F_x + \lambda \frac{\partial p}{\partial x} + \mu_u \Delta u^{m_u},$$

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = \rho_0 F_y + \lambda \frac{\partial p}{\partial y} + \mu_v \Delta v^{m_v},$$

$$\rho_0 \frac{\partial^2 w}{\partial t^2} = \rho_0 F_z + \lambda \frac{\partial p}{\partial z} + \mu_w \Delta w^{m_w},$$

$$p = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

with the initial conditions:

$$u|_{t=0} = d_u(\mathbf{r}), v|_{t=0} = d_v(\mathbf{r}), w|_{t=0} = d_w(\mathbf{r}),$$

$$\frac{\partial u}{\partial t}\Big|_{t=0} = d_{uu}(\mathbf{r}), \frac{\partial v}{\partial t}\Big|_{t=0} = d_{vv}(\mathbf{r}), \frac{\partial w}{\partial t}\Big|_{t=0} = d_{ww}(\mathbf{r})$$

and boundary conditions on the boundary S:

$$u|_S = q_u(\mathbf{r}), v|_S = q_v(\mathbf{r}), w|_S = q_w(\mathbf{r}),$$

In the integration region, a uniform grid is defined $\overline{\Omega}_h = \{x_i = ih_x, y_j = jh_y, z_k = kh_k, 0 \le i \le N_x, 0 \le j \le N_y, 0 \le k \le N_z\}$, and a time grid $\overline{\Omega}_\tau = \{t_n = n\tau, n = 0,1,..., N_\tau\}$.

Notation of grid functions: $f^n_{ijk} \equiv f(x_i, y_j, z_k, t_n)$.

Initial conditions are specified in internal nodes: $u^0_{ijk} = d_{uijk}, v^0_{ijk} = d_{vijk}, w^0_{ijk} = d_{wijk}$,

$$u^1_{ijk} = d_{uijk} + \tau d_{uuijk}, v^1_{ijk} = d_{vijk} + \tau d_{vvijk}, w^1_{ijk} = d_{wijk} + \tau d_{wwijk},$$

$$1 \le i \le N_x - 1, 1 \le j \le N_y - 1, 1 \le k \le N_z - 1$$

Explicit difference scheme:

$$Q^n_{uijk} = \mu_u[\frac{(u^n_{i-1jk})^{m_u} - 2(u^n_{ijk})^{m_u} + (u^n_{i+1jk})^{m_u}}{h_x^2} + \frac{(u^n_{ij-1k})^{m_u} - 2(u^n_{ijk})^{m_u} + (u^n_{ij+1k})^{m_u}}{h_y^2} +$$

$$+ \frac{(u^n_{ijk-1})^{m_u} - 2(u^n_{ijk})^{m_u} + (u^n_{ijk+1})^{m_u}}{h_z^2} + \rho_0 F_{xijk}],$$

$$Q^n_{vijk} = \mu_v[\frac{(v^n_{i-1jk})^{m_v} - 2(v^n_{ijk})^{m_v} + (v^n_{i+1jk})^{m_v}}{h_x^2} + \frac{(v^n_{ij-1k})^{m_v} - 2(v^n_{ijk})^{m_v} + (v^n_{ij+1k})^{m_v}}{h_y^2} +$$

$$+ \frac{(v^n_{ijk-1})^{m_v} - 2(v^n_{ijk})^{m_v} + (v^n_{ijk+1})^{m_v}}{h_z^2} + \rho_0 F_{yijk}],$$

$$Q^n_{wijk} = \mu_w[\frac{(w^n_{i-1jk})^{m_w} - 2(w^n_{ijk})^{m_w} + (w^n_{i+1jk})^{m_w}}{h_x^2} + \frac{(w^n_{ij-1k})^{m_w} - 2(w^n_{ijk})^{m_w} + (w^n_{ij+1k})^{m_w}}{h_y^2} +$$

$$+ \frac{(w^n_{ijk-1})^{m_w} - 2(w^n_{ijk})^{m_w} + (w^n_{ijk+1})^{m_w}}{h_z^2} + \rho_0 F_{zijk}],$$

$$\rho_0 \frac{u^{n+1}_{ijk} - 2u^n_{ijk} + u^{n-1}_{ijk}}{\tau^2} = Q^n_{uijk} + \lambda(\frac{u^n_{i-1jk} - 2u^n_{ijk} + u^n_{i+1jk}}{h_x^2} +$$

$$+ \frac{v^n_{i+1j+1k} - v^n_{i+1j-1k} - v^n_{i-1j+1k} + v^n_{i-1j-1k}}{4h_x h_y} + \frac{w^n_{i+1jk+1} - w^n_{i+1jk-1} - w^n_{i-1jk+1} + w^n_{i-1jk-1}}{4h_x h_z}),$$

$$\rho_0 \frac{v^{n+1}_{ijk} - 2v^n_{ijk} + v^{n-1}_{ijk}}{\tau^2} = Q^n_{vijk} + \lambda(\frac{v^n_{ij-1k} - 2v^n_{ijk} + v^n_{ij+1k}}{h_y^2} +$$

$$+ \frac{u^n_{i+1j+1k} - u^n_{i+1j-1k} - u^n_{i-1j+1k} + u^n_{i-1j-1k}}{4h_x h_y} + \frac{w^n_{ij+1k+1} - w^n_{ij+1k-1} - w^n_{ij-1k+1} + w^n_{ij-1k-1}}{4h_z h_y}),$$

$$\rho_0 \frac{w^{n+1}_{ijk} - 2w^n_{ijk} + w^{n-1}_{ijk}}{\tau^2} = Q^n_{wijk} + \lambda(\frac{w^n_{ijk-1} - 2w^n_{ijk} + w^n_{ijk+1}}{h_z^2} +$$

$$+\frac{u^{n}_{i+1jk+1}-u^{n}_{i+1jk-1}-u^{n}_{i-1jk+1}+u^{n}_{i-1jk-1}}{4h_xh_z}+\frac{v^{n}_{ij+1k+1}-v^{n}_{ij+1k-1}-v^{n}_{ij-1k+1}+v^{n}_{ij-1k-1}}{4h_zh_y}),$$

$$i=1,...,N_x-1, j=1,...,N_y-1, k=1,...,N_z-1$$

This scheme has a second-order error in all variables $O(\tau^2)+O(h_x^2)+O(h_y^2)+O(h_z^2)$. The stability of the scheme is ensured by the fulfillment of Courant's condition:

$$\frac{\tau^2\mu}{h_x^2+h_y^2+h_z^2}\le 1$$

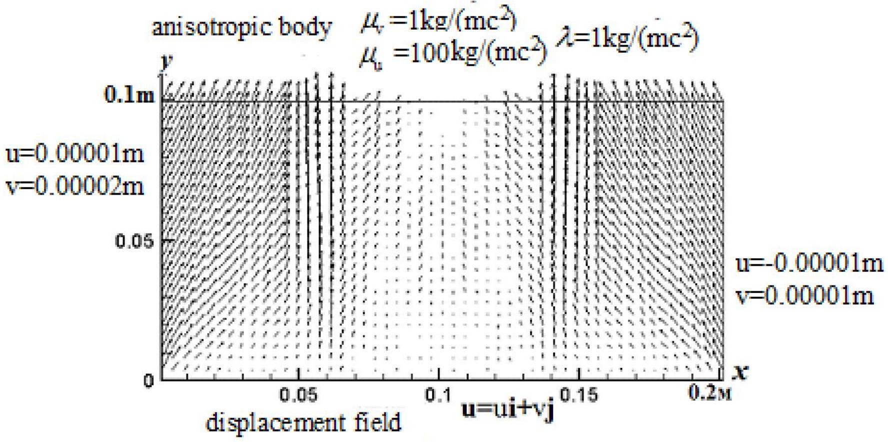


Figura 7- The field of displacements $\mathbf{u}=u\mathbf{i}+v\mathbf{j}$ i n an anisotropic body

Fig. 7 shows the displacement field according to Hooke's linear law $m_u=1, m_v=1$ in an anisotropic body, in Fig. 8 in an isotropic body.

On the upper plane of the bar, the Neumann boundary condition $\frac{\partial u}{\partial y}=0, \frac{\partial v}{\partial y}=0.$

Obviously, the difference in the coefficients leads to different displacement fields, hence to different fields of internal stresses.

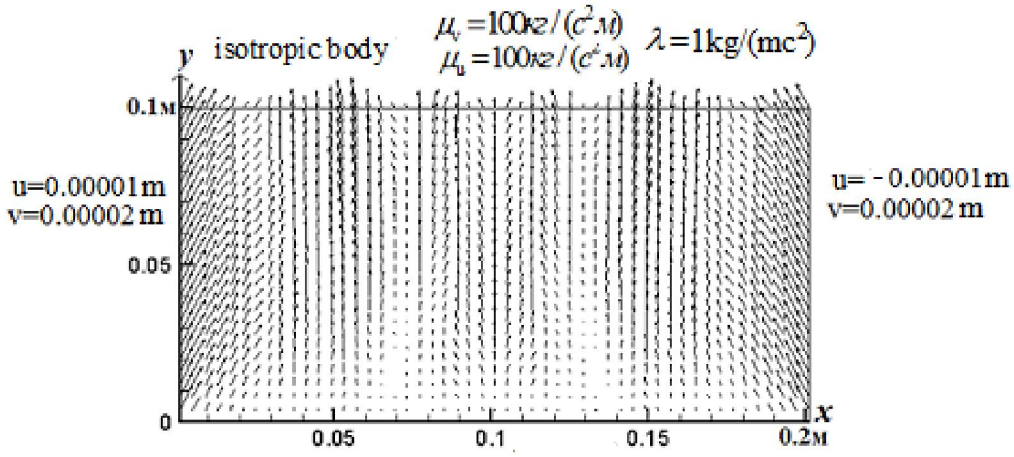


Figura 8 - The field of displacements in an isotropic body

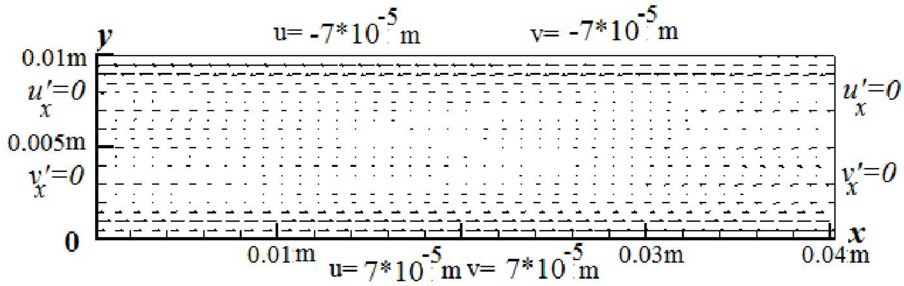


Figura 9 - Displacement field $\mathbf{u} = \mathrm{u}\mathbf{i} + \mathrm{v}\mathbf{j}$
by a nonlinear Hooke law of degree 3

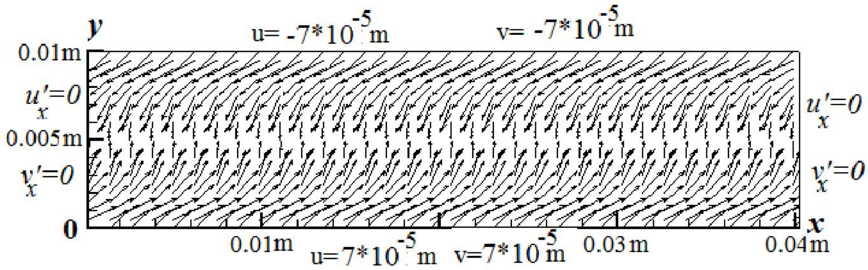


Figura 10 - Displacement field by a linear Hooke law

Fig. 9 shows the displacement field in an anisotropic body by Hooke's non-linear power law $m_u = 3, m_v = 3$. On the horizontal sides of the bar are given the Neumann boundary conditions are put on the lateral sides

$$\frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial x} = 0.$$

Lame coefficients in an anisotropic bar:

$$\lambda = 104.4\,\mathrm{kg}/(c^2\mathrm{m}), \mu_u = 80\,\mathrm{kg}/(c^2\mathrm{m}), \mu_v = 40\,\mathrm{kg}/(c^2\mathrm{m})$$

In Fig. 10 shows the displacement field in an anisotropic body according to Hooke's linear law $m_u = 1, m_v = 1$.

CONCLUSIONS

The physical conclusions of the normal and tangential stresses prove the asymmetry of the stress tensor in a solid deformed body both for the isotropic Hooke's law and for an anisotropic, including nonlinear one. Concrete examples of numerical calculations of the state of an elastic body show the inadequacy and inconsistency of the hypothesis of the symmetry of the stress tensor of a continuous medium and, accordingly, of the equations of the Lame elasticity theory.

The asymmetry of the stress tensor opens up wide possibilities for the modeling of displacements in a solid deformed body, which is shown by the application of Hooke's law in the anisotropic body of Fig. 7 and Hooke's nonlinear law of Fig. 9.

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СЫЗЫҚСЫЗ ГУКТЫҢ ЗАҢЫ БІРТЕКТЕС ЕМЕС ЖӘНЕ АНИЗОТРОПТЫҚ ДЕНЕЛЕРДІҢ СЕРПІЛІМДІК ТЕОРИЯСЫНДА

Аннотация. Тікелей сызықсыз Гук заңымен қатты майысқақ денелердің кернеулер тензорының компоненттері шығарылған. Ламенің екінші еселеушінің бағыттан тәуелділігі есептелген. Майысқақ қатты дененің серпілімдік теориясының кернеулер тензорының беттеспегендігі дәлелденген. Осыған сәйкес майысқақ қатты дененің сызықсыз серпілімдік теориясының теңдеулері жасалынған. Ламе гипотезасында толық емес жылжу дифференциалының беттескен жартысы қана пайдаланғаны көрсетілген, екінші антибеттескен жартысы лақтырылынған, соның салдарынан Ламе кернеулер тензорының беттескендігі шыққан. Жаңа теңдеулер үшін 2 ретті нақтылығы бар айқын схема жасалынған, соны пайдаланып жазық жолақтың серпілімдік күйі саналған, үстіңгі жақтауының ортасыны жанама кернеулер және тік кернеулер әсер еткенде. Дәл сондай схема Ламе теңдеулеріне де қолдалынған. Саналған жылжулардың үлестірулік суреттері салыстырынып жатқан теңдеулердің айырмашылықтарын бейнелейді және Ламе теңдеу- лерінің майысқақ қатты дененің күйіне сәйкес еместігін көрсетеді. Ламе теңдеулерінің жалғандығы теориялық және физикалық тұрпатта бекітілген.
Тірек сөздер: анизотроптылық, созылу, кернеулер, тензор, теңдеулер.

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НЕЛИНЕЙНЫЙ ЗАКОН ГУКА В ТЕОРИИ УПРУГОСТИ НЕОДНОРОДНЫХ И АНИЗОТРОПНЫХ ТЕЛ

Аннотация. Непосредственно из физической связи с нелинейным законом Гука выводятся компоненты тензора напряжений твердого деформируемого тела и новые нелинейные уравнения теории упругости с несимметричным тензором напряжений, как частный случай получаются уравнения с линейным законом Гука. Гипотеза Ламе и уравнения Ламе не имеют физической связи с законом Гука, в этом заключается их фальшивость. Ламе взял за основу приближенную формулу неполного дифференциала и предположил в своей гипотезе пропорциональность компонент тензора напряжений симметричной половине данного неполного дифференциала смещения, причем антисимметричная половина дифференциала отбрасывается, следствием чего является фальшивая симметричность тензора напряжений Ламе. Новые нелинейные уравнения аппроксимируются явной схемой, с применением которой численно рассчитано упругое состояние плоского бруска при действующих на верхней грани нормальном и касательном напряжениях. Такая же схема применена для уравнений Ламе. Полученные картины распределения смещений наглядно демонстрируют различие решений сравниваемых систем уравнений упругости, а также несоответствие решения уравнений Ламе данному состоянию деформируемого тела. Теоретически и физически подтверждена фальшивость уравнений Ламе.
Ключевые слова: растяжение, касательное, нормальное, напряжения, тензор.