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EMBEDDABILITY OF m-DEGREES INTO EQUIVALENCE RELATIONS IN THE ERSHOV HIERARCHY

Abstract. The paper is devoted to the study of equivalence relations in the hierarchy of Ershov. An equivalence relation R on ω is computably reducible to an equivalence relation S if there exists a computable function $f(x)$ such that for any $x$and y, the conditions xRy and $f(x)Sf(y)$ are equivalent. In this paper we construct isomorphic embeddings of semilattices of m-degrees into partial orders of equivalence relations in the hierarchy of Ershov with respect to computable reducibility.

Key words. Equivalence relations, computable reducibility, hierarchy of Ershov, computably enumerable sets, semilattice of computably enumerable m-degrees.

A set A is m-*reducible* toa set B (denoted by $A \leq_m B$) if there is a computable function f such that for any $x \in \omega$ the conditions $x \in A$ and $f(x) \in B$ are equivalent. Such a function f is called a reduction function. A set A is 1-*reducible* to a set B (denoted by $A \leq_1 B$) if $A \leq_m B$ and the corresponding reduction function is injective. A formula $A \equiv_m B$ means that $A \leq_m B$and $B \leq_m A$. The m-degree of a set A is denoted by $d(A)$, i.e. $d(A) = \{B: A \equiv_m B\}$. We use the same notation $\leq_m$ to denote a natural ordering on the set of m-degrees:

$$d(A) \leq_m d(B) \leftrightharpoons A \leq_m B.$$

A partial order $L_m^0 = (\{d(X): X \text{ is a recursively enumerable set and } X \neq \emptyset, \omega\}, \leq_m)$ is an upper semilattice and an ideal in the upper semilattice of all m-degrees. The supremum operation in L_m^0 is induced by the join of sets:

$$A \oplus B \leftrightharpoons \{2x: x \in A\} \cup \{2x+1: x \in B\}.$$

It is well-known that L_m^0 contains the greatest and the least elements. For further definitions and preliminaries on m-reducibility, we refer the reader to the monographs [1, 2].

Definition ([6]). A set A belongs to the class Σ_n^{-1} in the hierarchy of Ershov if there are computable functions $f(x,t)$ and $h(x,t)$ such that for any $x, t \in \omega$, the following conditions hold:

(1) $A(x) = \lim_s f(x,s)$ and $f(x,0) = 0$;

(2) $h(x,0) = n \;\&\; h(x,t+1) \leq h(x,t)$;

(3) $f(x,t+1) \neq f(x,t) \Rightarrow h(x,t+1) < h(x,t)$.

If a pair of functions $\langle f, h\rangle$ satisfies the conditions above, then we say that $\langle f, h\rangle$ is a Σ_n^{-1}-*approximation* of the set A. A set A lies in the class Π_n^{-1} in the hierarchy of Ershov if the complement of A belongs to the class Σ_n^{-1}. A set from the class Σ_n^{-1} (Π_n^{-1}) is also called a Σ_n^{-1}-set (Π_n^{-1}-set). Σ_1^{-1}-sets are known as computably enumerable sets. A detailed exposition of results on these sets can be found in [3, 4, 5, 6].

The classes Σ_n^{-1} and Π_n^{-1} are closed downwards under m-reducibility. It is known [3] that each of the classes contains a universal set. Moreover, the partial orders $(\Sigma_n^{-1}, \leq_m)$ and $(\Pi_n^{-1}, \leq_m)$ are upper semilattices.

Proposition 1. A set A belongs to Σ_n^{-1} if and only if there is a computable function $h(x,t)$ such that for any $x, t \in \omega$, the following holds:

(1) $A(x) = rest(\lim_s h(x,s), 2)$;

(2) $h(x,0) = 0 \,\& h(x,t) \leq h(x,t+1) \& \, h(x,t) \leq n$.

Proposition 2. A set A belongs to Π_n^{-1} if and only if there is a computable function $h(x,t)$ such that for any $x, t \in \omega$, the following holds:

(1) $A(x) = \overline{sg}(rest(\lim_s h(x,s), 2))$;

(2) $h(x,0) = 0 \,\& h(x,t) \leq h(x,t+1) \& \, h(x,t) \leq n$.

We assume that all considered sets and equivalence relations are defined on the domain ω. For a non-zero $n \in \omega$, Id_n is a computable equivalence relation which satisfies the following: $x \mathrm{Id}_n y$ if and only if x and y are equivalent modulo n. By Id we denote the identity equivalence relation. For an equivalence relation $E$and $a \in \omega$, $[a]_E$ denotes the E-equivalence class of the element a.

Definition. An equivalence relation E on ω is called aΣ_n^{-1}*equivalence relation* (*a* Π_n^{-1} *equivalence relation*) if the set E is a Σ_n^{-1}-set (Π_n^{-1}-set).

An equivalence relation R is *computably reducible* to an equivalence relation Q (denoted by $R \leq_c Q$) if there is a computable function f such that for any $x, y \in \omega$, the condition $(x,y) \in R$ holds if and only if $(f(x), f(y)) \in Q$; i.e. there is an algorithm which transforms different R-equivalence classes into different Q-equivalence classes. Equivalence relations R and Q and equivalent if each of them is reducible to the other one. The family of all equivalence relations which are equivalent to R is called the degree of an equivalence relation R.

It is clear that an equivalence relation E satisfies $E <_c Id$ if and only if $E \equiv_c \mathrm{Id}_n$ for some $n \in \omega$.

Definition (A. Sorbi and U. Andrews). An equivalence relation E is *dark* if E is incomparable with the identity equivalence relation under the reducibility $\leq_c$.

For an arbitrary c.e. set A, let $R_A = \{(x,y) \colon x = y \vee \{x,y\} \subseteq A\}$.

Proposition ([11]). Let A, B be non-empty c.e. sets.

1) $R_A$is computable if and only if A is computable.

2) $A \leq_1 B$implies that $R_A \leq_c R_B$.

3) If $R_A \leq_c R_B$, then $A \leq_m B$.

The proposition implies that c.e. 1-degrees are isomorphically embeddable into the structure of c.e. equivalence relations. It is well-known that c.e. 1-degrees do not form a semilattice. Hence, the structure of equivalence relations under computable reducibility is also not a semilattice.

In this work we study embeddings of semilattices of m-degrees into structures of equivalences in the hierarchy of Ershov. Results on embeddings of c.e. m-degrees into Rogers semilattices can be found in [7, 8, 9, 10]. For an embedding of c.e. 1-degrees into structures of equivalence relations, the reader is referred to [11, 12].

Embedding of semilattices of m-degrees into structures of equivalence relations in the hierarchy of Ershov.

Theorem 1. *For anyn* > 0, *the semilattice*$(\Sigma_n^{-1}, \leq_m)$*is isomorphically embeddable into the structure*$(\Pi_{2n}^{-1}$*equivalence relations,*$\leq_c)$.

Proof. We consider the following operator: for an arbitrary set X, we set

$$T(X) = \{(x,y) \colon \{x,y\} \subseteq X \vee \{x,y\} \subseteq \overline{X}\}.$$

It is clear that for any set X, the set $T(X)$ is an equivalence relation. We prove that the map $X \to T(X)$ induces an isomorphic embedding from the upper semilattice $(\Sigma_n^{-1}, \leq_m)$ into the structure $(\Pi_{2n}^{-1}$ equivalence relations,$\leq_c)$. We also show that our estimate of the level in the hierarchy of Ershov is sharp. In order to obtain this, we prove the following lemmas.

Lemma 1. If $X \in \Sigma_n^{-1}$, then $T(X) \in \Pi_{2n}^{-1}$.

Proof of Lemma 1. Suppose that a pair of functions $\langle f_X, h_X \rangle$ is a Σ_n^{-1}-approximation of the set X. We build an approximation of the set $T(X)$: for any $x, y \in \omega$, set

$$f\big((x,y),t\big) = |f_X(x,t) + f_X(y,t) - 1|;$$
$$h\big((x,y),t\big) = h_X(x,t) + h_X(y,t).$$

We prove that the pair $\langle f, h \rangle$ is a Π_{2n}^{-1}-approximation of the set $T(X)$.

1) $f\big((x,y),0\big) = |f_X(x,0) + f_X(y,0) - 1| = 1$; and

$\lim_s f\big((x,y),s\big) = |\lim_s f_X(x,s) + \lim_s f_X(y,s) - 1| = |X(x) + X(y) - 1|$.

The latter equation implies the following: $T(X)(x,y) = 1$ if and only if $X(x) = X(y)$. Therefore, $T(X)(x,y) = 1$ if and only if $\lim_s f\big((x,y),s\big) = 1$.

2) $h\big((x,y),0\big) = h_X(x,0) + h_X(y,0) = n + n = 2n$; and $h\big((x,y),t+1\big) = h_X(x,t+1) + h_X(y,t+1) \le h_X(x,t) + h_X(y,t) = h(x,y)$.

3) Suppose that $f((x,y),t+1) \ne f((x,y),t)$. Thus, either $f_X(x,t+1) \ne f_X(x,t)$, or $f_X(y,t+1) \ne f(y,t)$. Hence, either $h_X(x,t+1) < h_X(x,t)$, or $h_X(y,t+1) < h_X(y,t)$. In turn, this means that $h\big((x,y),t+1\big) = h_X(x,t+1) + h_X(y,t+1) < h_X(x,t) + h_X(y,t) = h\big((x,y),t\big)$.

Therefore, the pair of functions $\langle f, h \rangle$ is a Π_{2n}^{-1}-approximation of the set $T(X)$. Lemma 1 is proved.

Lemma 2. If $F \le_c T(X)$ for a Σ_n^{-1}-set X, then $F \equiv_c T(Y)$ for some Σ_n^{-1}-set Y.

Proof of Lemma 2. Suppose that for an arbitrary equivalence relation F, we have $F \le_c T(X)$ via a function f. Then the equivalence relation $T(X)$ contains at most two equivalence classes. Hence, the equivalence relation F also contains at most two classes. Therefore, if $Y = f^{-1}(X)$, then $F = T(Y)$. Lemma 2 is proved.

Lemma 3. $X \le_m Y$if and only if $T(X) \le_c T(Y)$.

Proof of Lemma 3. Both reductions can be realized by the same function. Lemma 3 is proved.

Lemma 4. For any Π_{2n}^{-1}-set A, there is a Σ_n^{-1}-set B such that $A \le_m T(B)$.

Proof of Lemma 4. Suppose that a pair of functions $\langle f_A, h_A \rangle$ is a Π_{2n}^{-1}-approximation of a set A. Moreover, let h_A be the function from Proposition 2. We build aΣ_n^{-1} approximation of a set B as follows:

$$f_B(2x,t) = \begin{cases} 1, rest(h_A(x,t),4) = 2; \\ 0, otherwise. \end{cases}$$
$$f_B(2x+1,t) = \begin{cases} 0, rest(h_A(x,t),4) = 0; \\ 1, otherwise. \end{cases}$$

$$\begin{cases} h_B(x,0) = n; \\ h_B(x,t+1) = h_B(x,t) - |f_B(x,t+1) - f_B(x,t)|. \end{cases}$$

It is not difficult to see that a pair of functions $\langle f_B, h_B \rangle$ is a Σ_n^{-1}-approximation of the set B. Furthermore, it is not hard to check that the reduction $A \le_m T(B)$ can be realized by the function $f(x) = (2x, 2x+1)$. Lemma 4 is proved.

Corollary 1. *If X is an m-complete Σ_n^{-1}-set, then $T(X)$ is an m-complete Π_{2n}^{-1}-set.*

Proof. Let X be an m-completeΣ_n^{-1}-set. We prove that any Π_{2n}^{-1}-set Asatisfies $A \le_m T(X)$. The proof of the theorem implies that there is a Σ_n^{-1}-set Y such that $A \le_m T(Y)$. It is clear that $T(Y) \le_c T(X)$. Suppose that that the reduction $T(Y) \le_c T(X)$ is realized by a function f; then the reduction $T(Y) \le_m T(X)$ is realized by the function

$$h\big((x,y)\big) = \big(f(x), f(y)\big).$$

Since m-reducibility is transitive, we have $A \le_m T(X)$.

Corollary 2. *For any non-computable set X, the equivalence relation $T(X)$ is dark.*

Corollary 3. *The semilattice of computably enumerable* m*-degrees is isomorphically embeddable into the structure* $(\Pi_2^{-1}$ *equivalence relations,* $\leq_c)$.

Corollaries 2 and 3 areevident.

Question. Is it possible to isomorphically embed the semilattice of c.e. m-degrees into the structure of c.e. equivalence relations?

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О ВЛОЖИМОСТИ - СТЕПЕНЕЙ В ОТНОШЕНИЯЭКВИВАЛЕНТНОСТИВ ИЕРАРХИИ ЕРШОВА

Аннотация. Работа посвящена исследованию отношений эквивалентности в иерархии Ершова. Отношение эквивалентности Rна ωвычислимо сводится к отношению эквивалентности S, если существует вычислимая функция $f(x)$, такая что, для любых xи y условия xRyи $f(x)Sf(y)$эквивалентны. В данной работе строятся изоморфные вложения полурешёток -степеней в частичные порядки отношений эквивалентности в иерархии Ершова относительно вычислимой сводимости.

Ключевые слова. Отношения эквивалентности, вычислимая сводимость, иерархия Ершова, вычислимо перечислимые множества, полурешетка вычислимо перечислимых m-степеней.

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ЕРШОВ ИЕРАРХИЯСЫНДА m-ДЕҢГЕЙЛЕРДІҢ ЭКВИВАЛЕНТТІК ҚАТЫНАСТАРҒА ЕНГІЗУЛЕРІ ТУРАЛЫ

Аннотация. Бұл мақала Ершов иерархиясындағы эквиваленттік қатынастарды зерттеуге бағышталған. ω жиынындаанықталғанR эквиваленттік қатынасы S эквиваленттік қатынасына есептелімді көшіріледі деп атаймыз, егер кез келген x және y элементтері үшін xRy және $f(x)Sf(y)$шарттарыэквивалент болатындай $f(x)$есептелімді функциясы табылатын болса. Бұл мақалада Ершов иерархиясындағы m-деңгейлерді есептелімді көшірулерге байланысты эквиваленттік қатынастардың жартлай ретіне изоморфты енгізулері құрылады.

Тірек сөздер. Эквивалеттік қатынастар, есептелімді көшірулер,Ершов иерархиясы, рекурсив саналымды жиындар, рекурсив саналымды m-деңгейлердің жатрыторы.