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## **PERTURBED ORBIT OF A CIRCULAR TYPE FOR THE HILL SECOND TASK**

**Annotation.** It is known that the Hill second intermediate orbit has found application in the theory of satellites motion [1, p.3]. Small parameter method has not yet been used in studies of the Hill second intermediate orbit.

This method in the [2, p.30] is given the following characteristics: “Method for constructing periodic solutions of nonlinear systems proposed by Henri Poincare has two features that should be borne in mind in practical use:

1. Procedure is effective only in the construction of periodic solutions of weakly nonlinear systems, as  $\mu$ -little.
2. The construction of each of the following approximation becomes harder than that of the first”. Of course, it is generally true.

This article shows that in solving nonlinear differential equations of the perturbed Hill circular problem, paragraph 2 becomes untenable. The reason is that the solutions of the second approximation have numerical coefficients, which have small order, which sharply reduces the amount of computation.

There is an opinion that the Poincare small parameter method does not characterize the evolution and oscillation of the perturbed orbits. This view was also wrong.

**Key words:** the Hill second task, perturbed orbit of a circular type, Poincare small parameter method, the angular frequency, free nonlinear salutation, oscillation, evolution.

Let us consider the plane perturbed Hill circular problem [1, c. 61]:

$$\left. \begin{aligned} \frac{d^2x}{dt^2} + \mu \frac{x}{r^3} = vx^3, & \quad \frac{d^2y}{dt^2} + \mu \frac{y}{r^3} = vy^3, \end{aligned} \right\} \quad (1)$$

where  $v$  – small parameter,  $r^2 = x^2 + y^2 = \text{const}$ .

We introduce the notation in (1)

$$k^2 = \frac{\mu}{r^3}, \text{ where } \mu - \text{const}, r^3 - \text{const}. \quad (2)$$

and apply the Poincaré small parameter method. To reduce the volume of the article, we will take the following steps:

1. The sequence of introducing series in degrees of small parameter with accuracy  $O(v^2)$  including and obtaining the corresponding systems of differential equations (1) will be performed according to the recommendations of M.I. Bat and others [3, p. 379].

In the problem there are two circular frequencies:  $k^2$  - the circular frequency of the unperturbed motion;  $p^2 = k^2 + \alpha_1 v + \alpha_2 v^2$  - the circular frequency of the perturbed motion, where  $\alpha_1$  and  $\alpha_2$  - the parameters determined from the condition for discarding resonant terms in particular solutions of systems of differential equations (1).

2. Everything related to the integration of linear homogeneous and linear inhomogeneous second-order differential equations with constant coefficients will be fulfilled in accordance with the well-known general theory (see textbooks on differential equations).

Completing p.1, we have a system of second-order differential equations with constant coefficients, the solutions of which with an accuracy  $O(v^2)$  inclusive, will be given by segments of series in powers of small parameter:

$$\left. \begin{aligned} x &= x_0 + vx_1 + v^2x_2 + O(v^3), \\ y &= y_0 + vy_1 + v^2y_2 + O(v^3). \end{aligned} \right\} \quad (3)$$

Thus, we have a system of differential equations corresponding to the first equation in (1):

$$\left. \begin{aligned} \ddot{x}_0 + p^2 \dot{x}_0 &= 0, \\ \ddot{x}_1 + p^2 x_1 &= \alpha_1 x_0 + x_0^3, \\ \ddot{x}_2 + p^2 x_2 &= \alpha_1 x_1 + \alpha_2 x_0 + 3x_0^2 x_1, \end{aligned} \right\} \quad (4)$$

and corresponding to the second equation of (1):

$$\left. \begin{aligned} \ddot{y}_0 + p^2 \dot{y}_0 &= 0, \\ \ddot{y}_1 + p^2 y_1 &= \alpha_1 y_0 + y_0^3, \\ \ddot{y}_2 + p^2 y_2 &= \alpha_1 y_1 + \alpha_2 y_0 + 3y_0^2 y_1. \end{aligned} \right\} \quad (5)$$

We integrate (4) and (5) under the following initial conditions:

$$\left. \begin{aligned} t = 0, \quad x_0(0) &= a, \quad x_1(0) = 0, \quad x_2(0) = 0, \quad \dot{x}_1(0) = 0, \quad \dot{x}_2(0) = 0, \\ y_0(0) &= 0, \quad y_1(0) = ap, \quad y_2(0) = 0, \quad \dot{y}_1(0) = 0, \quad \dot{y}_2(0) = 0. \end{aligned} \right\} \quad (6)$$

In accordance with p.1 and p.2, for the first equations from (4) and (5), taking into account (6), we have the zero approximation

$$x_0 = a \cos pt, \quad y_0 = a \sin pt, \quad (7)$$

where  $a$  – the radius of the unperturbed circular orbit, which corresponds to  $v = 0$ .

The solution of the second equations from (4) and (5) with regard for (6) has the form:

$$\left. \begin{aligned} x_1 &= \left( \frac{3a^3}{32p^2} \right) \cdot (\cos pt - \cos 3pt), \\ y_1 &= \left( \frac{3a^3}{32p^2} \right) \cdot (\sin 3pt - \sin pt), \end{aligned} \right\} \quad (8)$$

Here it is taken into account that

$$\begin{aligned} x_0 &= a \cos pt, \quad x_0^3 = a^3 \left( \frac{3}{4} \cos pt + \frac{1}{4} \cos 3pt \right), \\ y_0 &= a \sin pt, \quad y_0^3 = a^3 \left( \frac{3}{4} \sin pt - \frac{1}{4} \sin 3pt \right), \end{aligned}$$

and that the non-resonance condition of the first approximation gives:

$$\alpha_1 = -\frac{3}{4}a^2$$

The solution of the third equations from (4) and (5) under the initial conditions (6) has the form:

$$\left. \begin{aligned} x_2 &= \frac{9a^5}{3072p^2}(\cos 5pt - \cos pt), \\ y_2 &= \frac{9a^5}{512p^2}\left(\frac{13}{6}\sin pt - \sin 3pt + \frac{1}{6}\sin 5pt\right), \end{aligned} \right\} \quad (9)$$

Here it is taken into account that

$$\begin{aligned} 3y_0^2y_1 &= 3(a\sin pt)^2 \cdot \left(\frac{3a^3}{32p^2}\right) \cdot (\sin 3pt - \sin pt), \\ 3x_0^2x_1 &= 3(a\cos pt)^2 \cdot \left(\frac{3a^3}{32p^2}\right) \cdot (\cos pt - \cos 3pt), \end{aligned}$$

and that the non-resonance condition of the second approximation gives

$$\alpha_1 = \alpha_{2x} = \frac{9}{128} \cdot \frac{a^4}{p^2}, \quad (10)$$

$$\alpha_2 = \alpha_{2y} = \frac{-27}{128} \cdot \frac{a^4}{p^2}, \quad (11)$$

Here it affects  $x_0(0) = a$ ,  $y_0(0) = 0$ .

Recalling that

$$p^2 = k^2 + \alpha_1 v + \alpha_2 v^2$$

we have for the third equation in (4)

$$p^2 = k^2 + v \left( \frac{3}{4}a^2 \right) + v^2 \left( \frac{9}{128} \cdot \frac{a^4}{p^2} \right)$$

and accordingly for the third equation in (5)

$$p^2 = k^2 + v \left( \frac{3}{4}a^2 \right) - v^2 \left( \frac{27}{128} \cdot \frac{a^4}{p^2} \right),$$

Notably they have the same order of smallness  $O(v^2)$ . Since  $v$  is sufficiently small, we have  $O(v^2) \approx O(3v^2)$ .

Now we substitute in (3)  $x_0, y_0$  from (7),  $x_1$  and  $y_1$  from (8),  $x_2$  and  $y_2$  from (9) and find

$$\begin{aligned} x &= a\cos pt + v \left( \frac{3a^3}{32p^2} \right) \cdot (\cos pt - \cos 3pt) + v^2 \left( \frac{9a^5}{3072p^2} \right) \cdot (\cos 5pt - \cos pt), \\ y &= a\sin pt + v \left( \frac{3a^3}{32p^2} \right) \cdot (\sin 3pt - \sin pt) + v^2 \left( \frac{9a^5}{512p^2} \right) \cdot \left( \frac{13}{6}\sin pt - \sin 3pt + \frac{1}{6}\sin 5pt \right). \end{aligned} \quad (12)$$

According to the formula

$$r = \sqrt{x^2 + y^2} \quad (13)$$

It is possible to obtain the radius of the perturbed circular orbit

$$\rho = a \left\{ 1 + v \left( \frac{3a^2}{64p^2} \right) \cdot \left( \frac{1}{2} + \frac{3}{2} \cos 2pt - 2 \cos 4pt \right) \right\} + O(v^3). \quad (14)$$

At  $t = 0$ ,  $\rho = a$ , then we find  $\rho^{\min}$  and  $\rho^{\max}$ . For this we calculate  $\dot{\rho}$  and equate it to zero

$$3psin 2pt - 8psin 4pt = 0. \quad (15)$$

Let us find the critical points. Before we make a replacement

$$2pt = \tau, \quad 4pt = 2\tau,$$

then (15) will have the form

$$3 \sin \tau - 8 \sin 2\tau = 0 \quad (16)$$

or taking into account

$$\sin 2\tau = 2 \sin \tau \cos \tau \quad (17)$$

We have (17) instead of (16)

$$\sin \tau_1 (3 - 16 \cos \tau_2) = 0, \quad \tau_1 = 0^0, \quad \tau_2 = 79,2^0.$$

Now we need to check the critical points  $\tau_1$  and  $\tau_2$ , considering  $p - \text{const}$

$-1^0$	$0^0$	$5^0$
0,017	0	0,087
—	+	
min		

$70^0$	$79,2^0$	$90,1^0$
0,3420	0,1874	-0,002
+	—	
max		

The first critical point gives «min»

$$\tau_1 = 0, \quad t_1 = 0, \quad \rho^{\min} = a.$$

The second critical point gives «max»

$$\begin{aligned} \tau_2 &= 79 \frac{1}{5} = \left( \frac{396}{5} \right)^0, \quad \text{that corresponds to } t_2 = \frac{\tau_2}{2p} = \left( \frac{396}{10p} \right)^0; \\ \rho^{\max} &= a \left\{ 1 + v \left( \frac{3a^2}{64p^2} \right) \cdot \left( \frac{1}{2} + \frac{3}{2} \cos \tau_2 - 2 \cos 2\tau_2 \right) \right\} = a \left\{ 1 + v \left( \frac{a^2}{8p^2} \right) \right\}. \end{aligned}$$

Thus,  $\rho$  on the segment  $0 \leq \tau \leq \tau_2$  runs through all values of segment

$$a \leq \rho \leq a \left[ 1 + v \left( \frac{a^2}{8p^2} \right) \right].$$

The maximum of the radius of the perturbed circular orbit in time corresponds to  $t_2 = \frac{\tau_2}{2p}$ , and minimum -  $t_0 = 0$ .

It is seen from (14) that the perturbed radius consists of the sum of the evolutionary part of the radius  $a$  and oscillations with a very small amplitude  $\nu \left( \frac{3a^3}{64p^2} \right) \cdot \left( \frac{1}{2} + \frac{3}{2} \cos 2pt - 2 \cos 4pt \right)$ .

It may be affirmed, that the depicting point under the condition  $\frac{1}{2} + \frac{3}{2} \cos 2pt - 2 \cos 4pt = 0$  intersects a circle of the radius  $a$ , then under condition

$$\frac{1}{2} + \frac{3}{2} \cos 2pt - 2 \cos 4pt > 0$$

The representing point will be outside the evolutionary circle, and at fulfillment of the conditions

$$\frac{1}{2} + \frac{3}{2} \cos 2pt - 2 \cos 4pt < 0$$

will be inside this circle. Besides, the amplitude of deviations of the depicting point from the evolutionary circle will be different at each moment of time, it varies continuously.

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## ХИЛДЫҢ ЕКІНШІ ЕСЕБІНДЕГІ ҰЙЫТҚУЛЫ ШЕҢБЕР ТИПТЕС ОРБИТАЛАР

**Аннотация.** Хилдың екінші орталық орбитасы Жер серігінің қозғалыс теориясында қолданылып жатыр [1, б. 3].

Осы күнге дейін кіші параметр әдісі Хилдың орталық орбитасын зерттеуде қолданылмайды. Бұл әдіс туралы мынаңдай корытынды берілген [2, б. 30].

Әдісті қолданғанда төмендегі екі жайытты ұмытпандар:

1. Әдіс тек кіші параметрдің өте мардымсыз болғанында сәтті болады.

2. Әр келесі есептеу кадамы алдынғысынан күрделі болады. Жоғарғы жуықтауларда жұмыс көлемі көбее түседі.

Бұл макалада 2-ші ескерту кейде орындалмайтыны көрсетілген. Себебі есептеу барысында өте мардымсыз сандық коэффициенттер пайда болады да есептеу көлемін күрт қыскартады.

Және көшіліктің ойы бойынша бұл әдістің ұйытқулы орбиталардың эволюциясына, осцилляциясына ешқандай қатысы жоқ. Бұл ойда кате болып шықты.

**Түйін сөздер:** Хилдың екінші есебі, ұйытқулы шенберлік орбита, Пуанкаренің кіші параметрлер әдісі, шенберлік жүйлік, сыйықтық емес еркін қозғалыстар, осцилляция, эволюция.

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## **ВОЗМУЩЕННАЯ ОРБИТА КРУГОВОГО ТИПА ВО ВТОРОЙ ЗАДАЧЕ ХИЛЛА**

**Аннотация.** Известно, что вторая промежуточная орбита Хилла нашла приложения в теории движения спутников Земли [1, с.3]. Метод малого параметра еще не был использован при исследовании второй промежуточной орбиты Хилла. Этому методу в [2, с.30] дается следующая характеристика:

«Метод построения периодических решений нелинейных автономных систем, предложенный Анри Пуанкаре, обладает двумя особенностями, которые надо иметь в виду при практическом использовании:

1. Процедура эффективна только при построении периодических решений слабонелинейных систем, так как  $\mu$  – мало.

2. Построение каждого следующего приближения становится сложнее первого. Объем работ с каждым приближением растет лавинообразно».

Конечно, в общем случае эти предупреждения верны.

В данной статье показано, что при решении нелинейных дифференциальных уравнений возмущенной круговой задачи Хилла пункт 2 становится несостоятельным.

Причина состоит в том, что в решениях второго приближения появляются числовые коэффициенты, которые имеют малый порядок, что резко сокращает объем вычислений.

Бытует мнение, что метод малого параметра Пуанкаре никак не характеризует эволюцию и осцилляцию возмущенных орбит.

Это мнение тоже оказалось неверным.

**Ключевые слова:** вторая задача Хилла, возмущенная круговая орбита, метод малого параметра Пуанкаре, круговая частота, свободные нелинейные колебания, осцилляция, эволюция.