G. S. ABİLOVA, F. K. MUKANTAYEVA, V. V. VERBOVSKİY

(Suleyman Demirel University, Kaskelen, Republic of Kazakhstan)

SOME PROPERTIES OF FUNCTIONS DEFINABLE ON PARTIALLY ORDERED WEAKLY O-MINIMAL STRUCTURES

Abstract. The article surveys some topics related to o-minimality. A partially ordered structure is called weakly o-minimal if any definable subset is a finite union of convex sets. We consider some properties of functions definable on partially ordered weakly o-minimal structures. We show that there is no infinite interval such that each point of this interval is a point of a local minimum (maximum).

Keywords: partially ordered, o-minimality, definable functions, convex sets, local minimum (maximum).

Тірек сөздер: жартылай реттелген, о-минимальділігі, функция қолданушы, айқын жиынтық, локальды минимум (максимум).

Ключевые слова: частично упорядоченное, о-минимальности, пользователем функций, выпуклых множеств, локальный минимум (максимум).

In [1] van den Dries considered o-minimal expansions of the ordered field of reals. Later in [2–4] Anand Pillay and Charles Steinhorn introduced a general notion of o-minimality. After that Max Dickmann in [5] considered an example of a weakly o-minimal structure. And then Dugald Macpherson, David Marker, and Charles Steinhorn in [6] developed a theory of weakly o-minimal structures. Here we consider some generalization of this notion to partially ordered structures and investigates some properties of definable unary functions.

Recall that is subset A of a partially ordered structure M is called *convex* if for any two element a_1 and a_2 of A and any element b of M the condition $a_1 < b < a_2$ implies that b is an element of A. A maximal convex subset of A is called a *convex component* of A.

Definition (K. Kudaybergenov) A partially ordered structure is called *weakly o-minimal* if any definable subset is a finite union of convex sets.

Let $(M, <, f, \ldots)$ be a ordered set, and $(N, <)$ a totally ordered set, where $f\colon M \to N$ and the full induced structure on M is weakly o-minimal. That is if A is a definable subset of $M^n \times N^k$ in the structure $(M \cup N, <, f, \ldots)$ then the projection of A on M^n is definable in the full induced $(M, <, f, \ldots)$.

We define the following formulae:

$$\varphi_{>}(x, a) = (f(x) > f(a))$$

$$\varphi_{<}(x, a) = (f(x) < f(a))$$

$$\varphi_{=}(x, a) = (f(x) = f(a))$$

The intersection of $\varphi_{>}(M, a)$ with the interval (a, ∞) is definable, so there is a minimal convex components, because the number of convex components is finite. The same we can say for the intersections of $\varphi_{<}(M, a)$ and $\varphi_{=}(M, a)$ with the interval (a, ∞).

Note that the finitely many convex components of these three formulae have a as the left boundary point.

Definition. We say that a point a is of the type (k, m, n) from the right if there exist k convex components of $\varphi_{>}(M, a)$ with the left boundary point a, there are m convex components of $\varphi_{<}(M, a)$ with the left boundary point a and there are n convex components of $\varphi_{=}(M, a)$ with the left boundary point a.

Note, that similar things can be done for the intersections of these three formulae with the interval $(-\infty, a)$.

It is an simple exercise to write formulae $\Psi_{k,m,n}(x)$ and $\Theta_{k,m,n}(x)$ which express the fact that x is of the type (k, m, n) from the right and of the type (k, m, n) from the left, correspondingly.

Let $F_{h,i,j,k,m,n}(x)$ be the conjunction of $\Theta_{h,i,j}(x)$ and $\Psi_{k,m,n}(x)$.

Lemma 1 If $F_{h,i,j,k,m,n}(x)$ is true on an infinite interval and $j > 0$ or $n > 0$, then both j and k are equal to 0.

Proof. We consider only the case $j > 0$, because the other case is similar. Let $F_{h,i,j,k,m,n}(x)$ be true on (a,b). Let c belong to (a,b). Then there is d from (a,b) such that for any x from (c, d) it holds that $f(x) = f(c)$. Let e be from (d,c). Then $f(x) = f(c) = f(a) = f(e)$. Hence, $n > 0$.

Lemma 2 If $F_{0,0,j,0,0,n}(x)$ is true on an infinite interval (a,b), then the function f is constant on (a,b).

Proof is obvious.

Note that if the formula $F_{h,0,0,k,0,0}(a)$ is true, then a is a point of a local minimum. If the formula $F_{0,i,0,0,m,0}(x)$ is true, then a is a point of a local maximum.

Theorem 3 There is no infinite interval I such that each point of this interval is a point of a local minimum (maximum).

Proof. Assume the contrary, that such a function f does exist. Throughout the proof of the theorem all considered elements belong to the interval I.

Claim 1 We may assume that if $f(a) = f(b)$, then a and b are incomparable.

Proof of Claim1. Let $E(x, y)$ be defined as $f(x) = f(y)$. It is an equivalence relation. Consider $[a] = E(M, a)$. It contains no interval otherwise on this interval the formula $F_{h,i,j,k,m,n}(x)$ holds with $j > 0$ and $k > 0$.

There is a minimal convex component of the equivalence class $[a]$, because any finite partially ordered set has a minimal element, and this convex component is a point. Let $G(x) = (x$ is a minimal point of $[x])$. Note that $G(M)$ is infinite. If $G(M)$ contains no interval there is a minimal point a of $G(M)$ as its minimal convex component. Since I is open there is b from I such that $b < a$. Then any minimal element of $[b]$ is less than a, for a contradiction.

So we may assume for simplicity of notation that $G(M) = I$. Note that minimal elements of any partially ordered set are incomparable.

Notation U_a = the union of $\{ x > a : f(y) > f(x)$ for all y in $(a, x]\}$,
$\{ x < a : f(y) > f(x)$ for all y in $[x, a)\}$, and $\{a\}$.

That is the point a is a global minimum on U_a and U_a is a maximum convex set containing a with this property.

We denote a $<_U b$ iff U_a contains b, and $a \Diamond b$ iff either $a = b$ or $a <_U b$, or $a <_U b$.

Claim 2 1) U_a is a convex set.

2) if $a \neq b$, then $U_a \neq U_b$.

Proof is obvious.

Property 1 If the intersection of U_a and U_b is not empty, then either U_a is a subset of U_b or U_b is a subset of U_a, for any a, b with $a < b$.

Proof. Let the intersection of U_a and U_b be non-empty and $a < b$. Assume also that $f(a) < f(b)$. If b is in U_a then U_b is a subset of U_a.

Let b be not in U_a. Then there is d such that $a < d < b$ and $f(d) < f(a) < f(b)$. Since d is in U_b, so $U_a < d < U_b$. Then $U_a \cap U_b$ is empty, for a contradiction.

Property 2 The relation $<_U$ is a strict partial order.

Proof. Asymmetry and transitivity hold for $<_U$.

Property 3 For any chain $a_0 <_U a_1 <_U \ldots <_U a_n$ there is $a_{n+1} >_U a_n$.

Proof. Take a_{n+1} be an arbitrary element of U_b where $b = a_n$.

Property 4 Let $b <_U a$, $c <_U a$ and $b < c$. Then $b \Diamond c$.

Proof. Since $b <_U a$, $c <_U a$, then a is in $U_b \cap U_c$ and by Property 1 it holds that either U_b is a subset of U_c, or U_c is a subset of U_b.

Property 5 For any a the set $C_a = \{x : x <_U a\}$ does not contain infinite $<_U$-chain.

Proof. Assume the contrary that C_a contains an infinite chain. Then C_a contains an infinite interval J. Let d be in J and m, n is in $J \cap U_d$ be such that $m < d < n$.

By Property 4 it holds that $m \lozenge n$, say, $m <_U n$. Then n is in U_m. Since U_m is convex, so d is in U_m, that is $f(m) > f(d)$, for a contradiction.

Property 6 $<_U$ is a discrete order.

Property 7 For any a, c with $c <_U a$, there is b such that $c <_U b$ and $(a \lozenge b)$.

Proof. As b we take any element from U_c such that if $a > c$ then $b <$ c, and if $a < c$ then $b > c$.

Notation K is the set of all minimal elements respective to $<_U$

$$S(a) = \{x : a <_U x \text{ and there is no } y \text{ with } a <_U y <_U x\}$$

Property 8. Sets $S(a)$, where a runs over dom f, form a definable uniform partition of (dom f) $\setminus K$.

Proof. If $S(a) \cap S(b)$ is non-empty, then it contains c such that $a <_U c$, $b <_U c$. Then either $a <_U b$ or $b <_U a$. Then either $a < b < c$ and c is not in $S(a)$, or $b < a < c$ and c is not in $S(b)$, for a contradiction.

Property 9 K contains a minimal element.

Proof. Otherwise, it contains an infinite interval I. Let b be in I, and c in $U_b \cap I$. Then $c >_U b$, for a contradiction.

Property 10 For all a it holds that $S(a)$ is a subset of U_a.

Property 11 The set $S(a)$ is finite for all a.

Proof is similar to the proof of Property 9.

By partition K, $S(a)$, a runs over dom f, we construct an equivalence relation $E(x, y)$. Observe that each E-class contains a minimal element with respect to $<_U$. Properties 9 and 11 imply that E is an infinite equivalence with finite classes.

Let X consist of minimal elements of E-classes with respect to $<$. Then X is infinite. Let U be a maximal convex component of X. Let a be in U. By properties 7 and 3 there is $b <_U a$ such that b is not in X, $b>a$.

Let c be in U_b with $c>b$. Since $S(c)$ is not empty, it contains some d from X by property 3 and $S(c) > b$ by Property 10. Then $d > b > a$, both a and d belong to X, and b does not belong to X, for a contradiction.

The theorem is proved.

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Резюме

Г. С. Әбілова, Ф. К. Мұқантаева, В. В. Вербовский

(Сүлеймен Демирел атындағы университет, Қаскелен, Қазақстан Республикасы)

ЖАРТЫЛАЙ РЕТТЕЛГЕН ӘЛСІЗ О-МИНИМАЛЬДІ ҚҰРЫЛЫМДАРДА АНЫҚТАЛҒАН ФУНКЦИЯЛАРДЫҢ КЕЙБІР ҚАСИЕТТЕРІ

Мақалада о-минимальділіктің кейбір жалпы қорытындылары қарастырылған. Біз жартылай реттелген әлсіз о-минимальді құрылымдарда анықталған кейбір функциялардың қасиетін қарастырамыз. Біз интер-валдың әрбір нүктесі локальды минимум (максимум) нүктесі болатын шексіз интервалдың жоқ екенін көрсетеміз.

Тірек сөздер: жартылай реттелген, о-минимальділігі, функция қолданушы, айқын жиынтық, локальды минимум (максимум).

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Г. С. Абилова, Ф. К. Мукантаева, В. В. Вербовский

(Университет им. Сулеймана Демиреля, Каскелен, Республика Казахстан)

НЕКОТОРЫЕ СВОЙСТВА ФУНКЦИЙ, ОПРЕДЕЛИМЫХ
НА ЧАСТИЧНО УПОРЯДОЧЕННЫХ СЛАБО О-МИНИМАЛЬНЫХ СТРУКТУРАХ

В статье рассматривается некоторое обобщение о-минимальности. Мы рассматриваем некоторые свой-ства функций, определимых в частично упорядоченных слабо о-минимальных структурах. Мы показываем, что не существует бесконечного интервала, такого что каждая точка этого интервала есть точка локального минимума (максимума).

Ключевые слова: частично упорядоченное, о-минимальности, пользователем функций, выпуклых мно-жеств, локальный минимум (максимум).

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