A.A. ASHIMOV, YU.V. BOROVSKIY, A.S. AZHIBEKOVA

DEVELOPMENT AND APPLICATION OF THE PARAMETRICAL REGULATION THEORY OF THE SINGLE SECTOR ECONOMIC MODEL, TAKING INTO ACCOUNT THE LAG IN THE INPUT OF FUNDS

The paper presents some results on the development and application of elements of the parametrical regulation theory by example of one sector economic model taking into account the lag in the input of funds. We show application efficiency of the parametrical regulation theory for increasing the level of gross domestic product. In particular, optimal parameter values were obtained for the regulation of the economic system based on the considered mathematical model.

Keywords: investment lag, technical progress, production function, parametrical regulation, parametrical identification.

1 Introduction

The method for determining the optimal values of economic instruments of national economic development based on the Solow model using the approach of the parametrical regulation theory is proposed in [1]. Such an approach is the parametrical regulation theory of the dynamic systems development previously shown to be effective in some examples [2, 3].

In the classic Solow model it is assumed that investment are transformed into funds instantly. In the real economy, the application of capital investments is always going to lag. There are two known approaches to modeling the lag [4]. The first one assumes that lag is a fixed time lag τ, thus input of funds $V(t)$ at time t is investment made at the time $t-\tau$. The second approach is to use a distributed lag. It is assumed that investments of $I(\tau)$ made at the time τ will further be applied gradually, by shares. In this case, the dynamics of the input of funds, taking into account investments and the lag in the input of funds is as follows:

$$\frac{dV}{dt} = aI - aV.$$

In this paper we present new results on the development and application of the theory of parametrical regulation for finding optimal values of regulated parameters for the case single-sector economy model allowing for the lag in the input of funds, making it possible to improve the basic economic indicators.

2 Single-sector mathematical economic model, taking into account the lag in the input of funds

Adding the dynamic equation of the input of funds, taking into account investment and the lag in the input of funds to the standard Solow model, we get single sector economic model taking into account the lag in the input of funds. We write the mathematical model of the economy, taking into account the lag in the input of funds in the form of algebraic and differential equations in the following form:

$$\begin{cases} L(t) = L(0)e^{Vt} \\ K(t+1) = K(t) - \mu K(t) + V(t) \\ X(t) = AK(t)^{\alpha} L(t)^{1-\alpha} \\ I(t) = \rho X(t) \\ V(t+1) = V(t) + a(I(t) - V(t)) \end{cases} \quad (1)$$

with initial conditions:

$$K(0) = K_0,\ V(0) = V_0 \quad (2)$$

Here t – number of a month ($t = 0,1,2,\ldots$). Endogenous variables of the model: $X(t)$ – GDP (in billions of tenge per month); $L(t)$ – number of employed (in millions); $K(t)$ – basic funds (in billions of tenge); $I(t)$ – volume of investments (in billions of tenge per month). Exogenous variables: ν – monthly employment growth rate; μ – retired basic production assets; ρ – gross investments in GDP; A – coefficient of neutral technical progress; α – coefficient of elasticity for the funds; V_0 – volume of the funds at the initial time; $a = const$.

3. Estimates of parameters and mathematical model verification

Finding estimates of unknown values of the exogenous parameters of the researched macroeconomic model was carried out by solving the problem of parametrical identification applying the search methods in the sense of a minimum criterion that characterizes the deviation of the values of output variables of the model from the corresponding observed values based on the known statistics of the Republic of Kazakhstan for 2001-2007 [5].

To estimate the parameters A, α, μ, ρ, $V(0)$ we used the following criterion:

$$S^2 = \frac{1}{13}\sum_{i=0}^{6}\left[\left(\frac{X(t)-X^*(t)}{Y^*(t)}\right)^2 + \left(\frac{K(t)-K^*(t)}{K^*(t)}\right)^2\right] \to \min,$$

$$t = 12 * i \quad (3)$$

Here $t = 0$ corresponds to January of 2001; $L(t)$ – estimated level of employed people in billions of people; $X(t)$ – estimated GDP in billions of tenge per month; $K(t)$ – estimated basic funds in billions of tenge. The sign «*» corresponds to the observed values of corresponding variables.

The following constraints were imposed on the values of identifiable parameters: $-1 \le \nu \le 1$; $A > 0$; $0 \le \alpha \le 1$; $0 < \mu \le 1$; $0 < \rho \le 0.5$; $V_0 > 0$; $a > 0$.

As the result of the solution to the stated above problem of parametrical identification we obtained the following values of estimated parameters: $\nu = 0.0017$; $A = 0.07108$; $\alpha = 1$; $\mu = 7.86*10^{-17}$; $\rho = 0.4999$; $V_0 = 34.8974$; $a = 0.0299$. The relative value of standard deviation of calculated values of the variables from the corresponding observed values turned out to be approximately $100\,S = 2.9\%$.

The verification of the mathematical model was carried out with the help of retrospective forecasting of the model for the years of 2008-2009. Table 1 presents observed, calculated values and deviation of estimated values of basic endogenous output variables of the model from corresponding actual values.

In order to carry out the following experiments we repeatedly solve the problem of parametrical identification of the model for the period from 2001 till 2009. As the result of the problem solution of the parametrical identification of the model for the stated period of time we obtained the following values of the estimated parameters: $\nu = 0.0017$; $A = 0.0824$; $\alpha = 0.976$; $\mu = 0.0099$; $\rho = 0.4711$; $V(0) = 61.3908$; $a = 0.1298$. The value of the criterion of type (3) equals: $100S = 5.07\%$.

In computing experiments for finding minimum values of continuous function the Nelder-Mead algorithm of guided search [6] was applied.

4 Parametrical sensitivity analyses

The results of the conducted research on parametrical sensitivity of relative values for small deviations from obtained parameter values: ν, A, α, μ, ρ, $V(0)$, a of the mathematical model (2)-(3) are presented in Tables 2, 3. Each element of the matrix is calculated by the formula:

Table 1. **Results of retrospective forecasting of the model**

Year	December 2008	December 2009
$X^*(t)$, billions of tenge per month	1337.7	1341.6
$X(t)$, billions of tenge per month	1345.1	1724.1
Error (%)	0.55	28.5
$K^*(t)$, billions of tenge	17630.1	No data
$K(t)$, billions of tenge	18923.7	24254.3
Error (%)	7.34	-
$L^*(t)$, million people	7.857	7.903
$L(t)$, million people	7.752	7.915
Error (%)	-1.34	0.15

Table 2. **Sensitivity matrix when the parameter is increased by 1%**

Parameter \ Variable	L	K	X	I	V
v	0.1703	0.0040	0.0079	0.0079	0.0071
A	0.0000	2.1121	3.0816	3.0816	2.9275
б	0.0000	16.1201	25.1893	25.1893	23.5757
μ	0.0000	-0.7807	-0.7621	-0.7621	-0.7096
c	0.0000	2.1121	2.0610	3.0816	2.9275
a	0.0000	0.3166	0.3090	0.3090	0.4120
V_0	0.0000	0.0890	0.0868	0.0868	0.0872

$$F(E) = \frac{\Pi(T) - \Pi^*(T)}{0.01\Pi^*(T)}.$$

Here T – period of time in months (T = 96); $\Pi^*(T)$ – value of endogenous variable, obtained when running the model with the values of exogenous parameters obtained from the preliminary estimates of the parameters; $\Pi(T)$ – value of corresponding endogenous variable, obtained when increasing (decreasing) the variable exogenous parameter E by 1%, while the values of other exogenous parameters are set equal to the results of their preliminary estimation.

The analysis of the sensitivity matrix shows that when exogenous indicators increase by 1%, the indicators grow up, the negative effect is observed only when the retired basic production assets value increases

5 The choice of optimal laws of parametrical regulation

Let us consider the possibility of carrying out the effective state policy through finding the optimal laws of regulation by the example economic parameter – retired basic production assets (ρ).

The choice of optimal laws of parametrical regulation was made in the framework of the following dependencies:

$$U_1(t) = \rho^* + k_1 \frac{K_1(t) - K(0)}{K(0)};$$

$$U_2(t) = \rho^* - k_2 \frac{K_2(t) - K(0)}{K(0)};$$

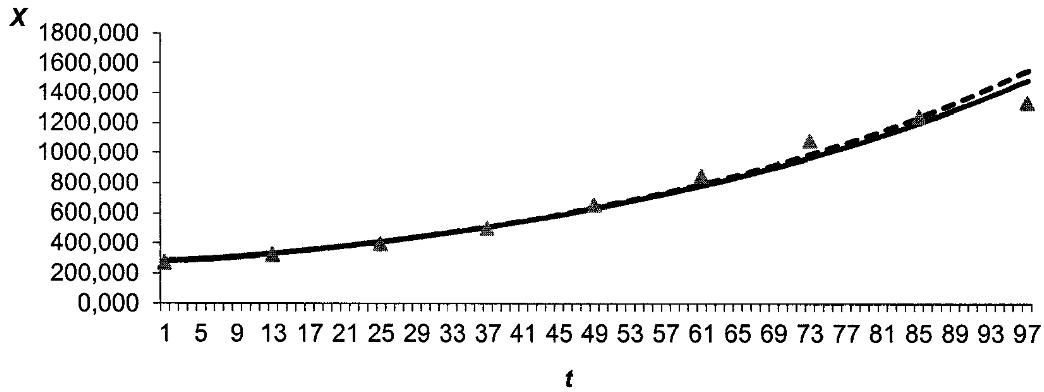
$$U_3(t) = \rho^* + k_3 \frac{X_3(t) - X(0)}{X(0)};$$

$$U_4(t) = \rho^* - k_4 \frac{X_4(t) - X(0)}{X(0)};$$

$$U_5(t) = \rho^* + k_5 \frac{V_5(t) - V(0)}{V(0)};$$

$$U_4(t) = \rho^* - k_6 \frac{V_6(t) - V(0)}{V(0)} \quad (4)$$

Here U_i – i-th law of parameter regulation ρ ($i = \overline{1,6}$); k_i – adjustable coefficient of i-th law of regulation, $k_i > 0$; ρ^* – constant, equal to the basic value of the parameter ρ; $K_i(t)$, $X_i(t)$, $V_i(t)$ –



—— without regulation; - - - with regulation; —▲— observed values

Figure 1. GDP without and with parametrical regulation of the parameter ρ

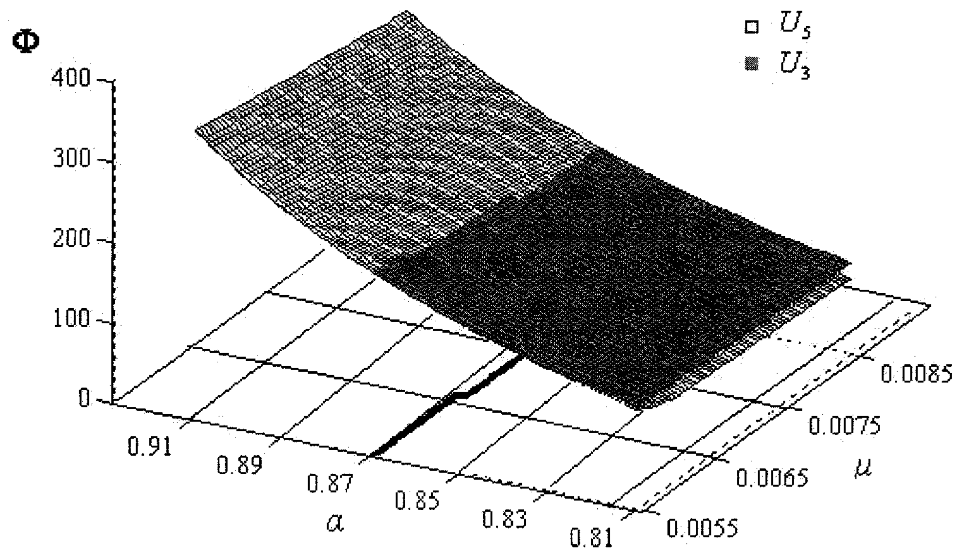


Figure 2. Criterion Φ optimal values graphs

solution of the problem (1) with initial condition $K_i(0) = K(0)$ when the law of regulation U_i is applied. Application of the law of regulation U_i implies substitution of the function from the right sides of the system (4) to the system (1) instead of parameter ρ; $t = 0$ – starting time of regulation (year 2001).

The problem of searching for optimal law of parametrical regulation on the level of one of economic parameters ρ can be formulated in the following way. The problem of choice of the parametrical regulation law on the level of one of economic parameters ρ can be formulated in the following way. Based on the mathematical model (1)-(2) find optimal law of parametrical regulation U_i and its coefficient k_i in the framework of algorithms (4), which would provide the maximum of the criterion:

$$\Phi = \frac{1}{97}\sum_{t=0}^{96} X_i(t) \rightarrow \max \qquad (5)$$

Under constraints:

$$0 \le \rho_i(t) \le 0.5, \; i = \overline{1,6} \qquad (6)$$

The formulated problem is solved in two steps:

- At the first step, we define optimal values of coefficients the optimal values of the coefficients k_i for each U_i law by trying their values in the range $[0;1]$, quantized with sufficiently small step to ensure Φ criterion maximum under the constraints (6);

- At the second step we choose the law of ρ parameter optimal regulation based on the results of the first stage by the maximum value of the criterion Φ.

As a result of numerical solution of the first step of the set problem on application of scenario approach

Table 3. **Numerical solution results of the first stage of the problem of optimal law choice of parametrical regulation**

Law	Coefficient optimal value	Φ criterion value
U_1	0.00617	726.0486
U_2	0	715.0072
U_3	0.00645	726.1818
U_4	0	715.0072
U_5	0.00289	729.4744
U_6	0	715.0072

to the evolution of economic system we obtained the criterion values for each scenario from the given above set (4). The numerical solution of the first phase of the task on a scenario approach to the evolution of the economic system were found to be the criterion for each of the scenarios presented above, a set of (4). The criterion values and the coefficient corresponding to each scenario are shown in Table 3.

According to the results of calculations we can conclude, that optimal in sense of a criterion (6) is a scenario U_5. In this case the value of Φ criterion equals to 729,4744. The value of the corresponding coefficient is $k_5 = 0.00289$.

The values of endogenous variable X (GDP) of the model without ρ parameter regulation, applying the scenario U_5 are given in the graphical view on the Figure 1.

6 Finding the bifurcation points

Consider the possibility of finding the bifurcation points for the extremals of calculus variation problem on the choice of the law of parametrical regulation of the model of economic growth under consideration.

Consider the dependence of results of the choice of parametrical regulation law on the level of the parameter ρ on uncontrolled parameters (μ, α) which values belong to the rectangular area Λ:

$$\alpha \in [0.81; 0.91], \ \mu \in [0.0055; 0.009].$$

As a result of the computing experiment we obtained the dependence graphs of the criterion's (5) Φ optimal value on values of parameters (α, μ) for each of the six possible laws U_i, $i = \overline{1,6}$. Figure 2 presents the graphs for the laws U_5 and U_3, which provide the criterion's maximum values on the considered domain Λ of the parameter values (α, μ), intersection line of the corresponding surfaces and the projection of this intersection line on the plane of parameter values (α, μ), consisting of the bifurcation points of extremals. This projection divides a rectangle Λ into two parts, one of which contains the optimal control law U_5, and the other – U_3. At the very projection of the line both of the laws are optimal.

According to this dependence analysis of the considered variation calculus problem's results on values of uncontrolled parameters (α, μ), the choice of optimal laws of parametrical regulation can be approached as follows. If parameter values (α, μ) are left to the line of bifurcation in the rectangle Λ (Figure 2), then the law U_5 can be recommended as an optimal law, if parameter values (α, μ) are right to the bifurcation line in the rectangle Λ, then the law U_3 is recommended as an optimal law. If parameter values are on the line of bifurcation, then any of the two laws U_3, U_5 can be recommended as an optimal law.

7 Models' structural stability analysis

Investigation of robustness (of structural stability) of the model (1), (2) is based on finding the estimates of the chain-recurrent set $R(f, N)$ of a dynamical system in a compact set N of its phase space. The parallelepiped of its phase space, including all possible trajectories of the economic system evolution was taken as a compact N for the mathematical model of economic system for the considered period of time. We determine f mapping, defined in N and by a shift along the trajectories of a dynamical system for a fixed period of time. Next, we construct a C partition of N on cells N_i. We set the directed graph G, whose vertices correspond to the cells, and the edges connecting the cell N_i with N_j comply with the terms of intersection of the image of a single cell $f(N_i)$ with another cell N_j. The graph G contains all return vertices (tops belonging to cycles). If the set of vertices is empty, then $R(f, N)$ is empty and the process of its localization is completed. The conclusion about a weak structural stability of dynamical system [7] is made.

The compact N in the phase space of the variables of the considered model (1)-(2) K (basic funds), L (number of employed) was defined with the help of inequalities:

$$1000 \le K \le 22000, \; 5 \le L \le 10.$$

As the result of the computing experiment we obtained the result of $R(f, N) = \varnothing$. This implies that the Solow single-sector model of the economic growth allowing for the lag in the input of funds for describing the interaction of the goods market and the money market is assessed as weakly structurally stable in the compact N.

8 Conclusion

1. We show the application efficiency of the parametrical regulation theory by the example of the Solow mathematical model allowing for the lag in input of funds. Based on the considered mathematical model, optimal values of regulated parameters of economic policy were proposed.

2. New optimal laws of parametrical regulation of world dynamics development on the level of one economic parameter were found based on the parametrical regulation theory methods.

3. Bifurcation points were found for a given range of uncontrolled parameters.

4. Robustness of the Solow mathematical model allowing for the lag in the input of funds was set.

5. The obtained results can be recommended to be used in the in the development and implementation of effective public policy.

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Резюме

Осы жұмыста негізгі қорларды енгізудің кешігуін ескеретін бір секторлы экономикалық моделінің нұсқасында параметрлік басқару теориясын қолдану мен дамуға қатысты кейбір нәтижелер берілген. Параметрлік реттеу теориясын жалпы ішкі өнімді өсіруге қолданудың тиімділігі көрсетілген. Атап айтқанда, қарастырылып отырған математикалық модель негізінде экономикалық жүйенің дамуын реттейтін параметрлердің оңтайлы мәндері табылған.

Резюме

В работе представлены некоторые результаты по развитию и применению элементов теории параметрического регулирования на примере односекторной модели экономики с учётом запаздывания во вводе фондов. Показана эффективность применения теории параметрического регулирования для увеличения валового внутреннего продукта страны. В частности, получены оптимальные значения параметров для регулирования развития экономической системы на базе рассмотренной математической модели.

Kazakh National Technical University named after K. Satpayev, Almaty

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