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F. B. BELISAROVA, P. YU. TSYBA, I. I. KULNAZAROV, SH. MYRZAKUL, K. K.YERZHANOV

EXACT SOLUTION OF G-ESSENCE

The g-essence model is studied. It is shown that this model admits some important physical reductions namely k-essence and f-essence. The exact solution of the g-essence model is constructed. This solution can describes the observed accelerated expansion of the universe.

More than ten years after its initial discovery [1-2], cosmic acceleration remains an unsolved problem. In fact, this phenomenon is so much at odds with conventional particle physics and cosmology that a solution might require a complete reformulation of the laws of physics governing both very small scales and very large scales. The contemporary models trying to explain cosmic acceleration using quantum field theory and general relativity fail to provide a convincing framework. The observational evidence from different sources for the present stage of accelerated expansion of our universe has driven the quest for theoretical explanations of such feature. More generally, modern theoretical physics are faced with two severe theoretical difficulties, that can be summarized as the dark energy and the dark matter problems. Several theoretical models, responsible e.g. for the accelerated expansion, have been proposed in the literature, in particular, models with some sourses and modified gravity, amongst others.

During last years theories described by the action with the non-canonical kinetic terms, k-essence, attracted a considerable interest. Such theories were first studied in the context of kinflation [3], and then the k-essence models were suggested as dynamical dark energy for solving the cosmic coincidence problem [4-6].

In the recent years several approaches were made to explain the accelerated expansion by choosing fermionic fields as the gravitational sources of energy (see e.g. refs. [7-24]). In particular, it was shown that the fermionic field plays very important role in: i) isotropization of initially anisotropic spacetime; ii) formation of singularity free cosmological solutions; iii) explaining latetime acceleration. Note that the formulation of the gravity-fermionic theory has been discussed in detail elsewhere [25-28].

Quite recently, the fermionic counterpart of the scalar k-essence was presented in [23] and called f-essence for short. In the present paper, we explore the so-called g-essence for the particular case $L = R + 2\left[\alpha X^n + \varepsilon Y - V(\bar{\psi}, \psi)\right]$. In particular, the exact solution of the model is found.

Let us start with the M_{34} - model. It has the following action [23]

$$S = \int d^4x\sqrt{-g}\left[R + 2K(X, Y, \phi, \psi, \bar{\psi})\right], \qquad (1)$$

where K is some function of its arguments, ϕ is a scalar function, $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ is a fermionic function and $\bar{\psi} = \psi^+\gamma^0$ is its adjoint function. Here

$$X = 0{,}5g^{\mu\nu}\nabla_\mu\phi\nabla_\mu\phi,$$
$$Y = 0{,}5i\left[\bar{\psi}\Gamma^\mu D_\mu\psi - (D_\mu\bar{\psi})\Gamma^\mu\psi\right] \qquad (2)$$

are the canonical kinetic terms for the scalar and fermionic fields, respectively. ∇_μ and D_μ are the covariant derivatives.

The variation of the action (1) with respect to $g_{\mu\nu}$ gives us the following energy-momentum tensor for the g-essence fields:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} =$$
$$= K_X\nabla_\mu\phi\nabla_\nu\phi + 0{,}5iK_Y\left[\bar{\psi}\Gamma_{(\mu}D_{\nu)}\psi - D_{(\mu}\bar{\psi}\Gamma_{\nu)}\psi\right]$$
$$- g_{\mu\nu}K = 2K_X X u_{1\mu}u_{1\nu} + K_Y Y u_{2\mu}u_{2\nu} - Kg_{\mu\nu}, \qquad (3)$$

where $K_X = \partial K/\partial X$, $K_Y = \partial K/\partial Y$, $u_{1\mu} = \nabla_\mu\phi/\sqrt{2X}$ etc. The equation of motion for the scalar field ϕ is obtained by variation of the action (1) with respect to ϕ,

$$-\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\phi}=$$
$$=\left(K_X g^{\mu\nu}+K_{XX}\nabla^\mu\phi\nabla^\nu\phi\right)\nabla_\mu\nabla_\nu\phi+2XK_{X\phi}-K_\phi . \quad (4)$$

Varying the action (1) with respect to $g_{\mu\nu}$ gives us the Einstein equations

$$-\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}=R_{\mu\nu}-0{,}5Rg_{\mu\nu}-T_{\mu\nu}=0\,, \quad (5)$$

where $R_{\mu\nu}$ is the Ricci tensor. Similarly, from the Euler-Lagrange equations applied to the Lagrangian density K we can obtain the Dirac equations for the fermionic field ψ and its adjoint $\bar\psi$ coupled to the gravitational field.

With the general formaism described above, we are now interested to investigate cosmology. Let us consider the flat FRW universe, in which the metric is given by

$$ds^2=dt^2-a\left(dx^2+dy^2+dz^2\right), \quad (6)$$

and the vierbein is chosen to be

$$\left(e^\mu_a\right)=diag(1,1/a,1/a,1/a),\quad \left(e^a_\mu\right)=diag(1,a,a,a). \quad (7)$$

In the case of the FRW metric (6), the equations corresponding to the action (1) look like

$$3H^2-\rho=0\,, \quad (8)$$
$$2\dot H+3H^2+p=0\,, \quad (9)$$
$$K_X\ddot\phi+\left(\dot K_X+3HK_X\right)\dot\phi-K_\phi=0\,, \quad (10)$$
$$K_Y\dot\psi+0{,}5\left(3HK_Y+\dot K_Y\right)\psi-i\gamma^0K_{\bar\psi}=0\,, \quad (11)$$
$$K_Y\dot{\bar\psi}+0{,}5\left(3HK_Y+\dot K_Y\right)\bar\psi-iK_\psi\gamma^0=0\,, \quad (12)$$
$$\dot\rho+3H(\rho+p)=0\,, \quad (13)$$

where the kinetic terms, the energy density and the pressure take the form

$$X=0{,}5\dot\phi^2,\quad Y=0{,}5i\left(\bar\psi\gamma^0\dot\psi-\dot{\bar\psi}\gamma^0\psi\right) \quad (14)$$

and

$$\rho=2K_XX+K_YY-K,\quad p=K\,. \quad (15)$$

Note that for the FRW metric (6), the equations of the M_{34} - model (1) can be rewritten as

$$3H^2-\rho=0\,, \quad (16)$$
$$2\dot H+3H^2+p=0\,, \quad (17)$$
$$\left(a^3K_X\dot\phi\right)_t-a^3K_\phi=0\,, \quad (18)$$
$$\left(a^3K_Y\psi_j^2\right)_t-2iK_{\bar\psi}\left(\gamma^0\psi\right)_j=0\,, \quad (19)$$
$$\left(a^3K_Y\psi_j^{*2}\right)_t+2iK_\psi\left(\bar\psi\gamma^0\right)_j=0\,, \quad (20)$$
$$\dot\rho+3H(\rho+p)=0\,. \quad (21)$$

Finally we present the following useful formula

$$K_YY=0{,}5iK_Y\left(\bar\psi\gamma^0\dot\psi-\dot{\bar\psi}\gamma^0\psi\right)=-0{,}5\left(K_\psi\psi+K_{\bar\psi}\bar\psi\right) \quad (22)$$

and the equation for $u=\bar\psi\psi$

$$\left[\ln\left(ua^3K_Y\right)\right]_t u=-iK_Y^{-1}\left(\bar\psi\gamma^0K_{\bar\psi}-K_\psi\gamma^0\psi\right). \quad (23)$$

Let us consider the purely kinetic case of the M_{34} - model that is when $K = K(X,Y)$. In this case, the system (8)-(13) becomes

$$3H^2-\rho=0\,, \quad (24)$$
$$2\dot H+3H^2+p=0\,, \quad (25)$$
$$a^3K_X\dot\phi-\sigma=0\,, \quad (26)$$
$$a^3K_Y\psi_j^2-\varsigma_j=0\,, \quad (27)$$
$$a^3K_Y\psi_j^{*2}-\varsigma_j^*=0\,, \quad (28)$$
$$\dot\rho+3H(\rho+p)=0\,, \quad (29)$$

where $\sigma(\varsigma)$ is the real (complex) constant. Hence we immediately get the solutions of the Klein-Gordon and Dirac equations, respectively, as

$$\phi=\sigma\int\frac{dt}{a^3K_X},\quad \psi_j=\sqrt{\frac{\varsigma_j}{a^3K_Y}}\,. \quad (30)$$

Also the following useful formula takes place

$$X=\frac{0{,}5\sigma^2}{a^6K_X^2}\quad or\quad K_X=\frac{\sigma}{a^3\sqrt{2X}}\,. \quad (31)$$

It is interesting to note that for the purely kinetic g-essence the solutions of the Klein-Gordon and Dirac equations are related by the formula

$$\dot\phi=\sigma\varsigma_j^{-1}\psi_j^2\,. \quad (32)$$

Let us conclude this section as: for the purely kinetic case $K = K(X,Y)$ from (22) follows that $Y = 0$ so that in fact we have $K = K(X,Y) = K(X,0) = K(X)$. So we will go further, having passed by this case.

Let us now we consider the following particular case of the M_{34} - model (1):

$$K=K_1=K_1(X,\phi). \quad (33)$$

Then the system (8)-(13) takes the form of the equations of the k-essence

$$3H^2-\rho_k=0, \quad (34)$$

$$2\dot{H}+3H^2+p_k=0, \quad (35)$$

$$K_{1X}\ddot{\phi}+\left(\dot{K}_{1X}+3HK_{1X}\right)\dot{\phi}-K_{1\phi}=0, \quad (36)$$

$$\dot{\rho}_k+3H\left(\rho_k+p_k\right)=0, \quad (37)$$

where the energy density and the pressure are given by

$$\rho_k=2K_{1X}X-K_1, \quad p_k=K_1. \quad (38)$$

It is interesting to note that in the case of the FRW metric (6), k-essence and F(T) – gravity (modified teleparallel gravity) are eqivalent to each other, if $a=e^{\pm\frac{\phi-\phi_1}{\sqrt{12}}}$.

Now we consider the following reduction of the M_{34} - model (1):

$$K=K_2=K_2\left(Y,\psi,\bar{\psi}\right). \quad (39)$$

that corresponds to the M_{33} - model that is the f-essence [23]. In this case, the system (8)-(13) becomes

$$3H^2-\rho_f=0, \quad (40)$$

$$2\dot{H}+3H^2+p_f=0, \quad (41)$$

$$K_{2Y}\dot{\psi}+0{,}5\left(3HK_{2Y}+\dot{K}_{2Y}\right)\psi-i\gamma^0K_{2\bar{\psi}}=0, \quad (42)$$

$$K_{2Y}\dot{\bar{\psi}}+0{,}5\left(3HK_{2Y}+\dot{K}_{2Y}\right)\bar{\psi}-iK_{2\psi}\gamma^0=0, \quad (43)$$

$$\dot{\rho}_f+3H\left(\rho_f+p_f\right)=0, \quad (44)$$

where

$$\rho_f=K_{2Y}Y-K_2, \quad p_f=K_2. \quad (45)$$

Let us we present some solution of the g-essence. To do it, we consider the case

$$K=K\left(X,Y,\psi,\bar{\psi}\right)=\alpha X^n+\varepsilon Y-V\left(\psi,\bar{\psi}\right). \quad (46)$$

We have

$$X=\sqrt[2n-1]{\frac{\sigma^2}{2n^2\alpha^2a^6}}, \quad (47)$$

$$Y=-2\varepsilon^{-1}\left[\dot{H}+\alpha n\left(\frac{\sigma^2}{2n^2\alpha^2a^6}\right)^{\frac{n}{2n-1}}\right], \quad (48)$$

$$V=3H^2-(2n-1)\alpha\left(\frac{\sigma^2}{2n^2\alpha^2a^6}\right)^{\frac{n}{2n-1}}, \quad (49)$$

$$K=-2\dot{H}-3H^2. \quad (50)$$

Now we would like to construct some solutions. Consider examples. Let consider the power-law solution

$$a=a_0t^\lambda. \quad (51)$$

Then we get

$$X=\sqrt[2n-1]{\frac{\sigma^2}{2n^2\alpha^2a_0^6t^{6\lambda}}}, \quad (52)$$

$$Y=-2\varepsilon^{-1}\left[-\frac{\lambda}{t^2}+\alpha n\left(\frac{\sigma^2}{2n^2\alpha^2a_0^6t^{6\lambda}}\right)^{\frac{n}{2n-1}}\right], \quad (53)$$

$$V=\frac{3\lambda^2}{t^2}-(2n-1)\alpha\left(\frac{\sigma^2}{2n^2\alpha^2a_0^6t_0^{6\lambda}}\right)^{\frac{n}{2n-1}}, \quad (54)$$

$$K=\frac{\lambda(2-3\lambda)}{t^2}. \quad (55)$$

We now consider the potential of the form $V=V(u)$. Then from (23) follows that

$$u=\frac{c}{\varepsilon a^3}, \quad Y=\varepsilon^{-1}V_uu. \quad (56)$$

As

$$u=\frac{c}{\varepsilon a_0^6t^{3\lambda}}, \quad t=\left[\frac{c}{\varepsilon a_0^6u}\right]^{\frac{1}{3\lambda}}, \quad (57)$$

the expression for the potential takes the form

$$V=3\lambda^2\left(\frac{\varepsilon a_0^3u}{c}\right)^{\frac{2}{3\lambda}}-(2n-1)\alpha\left(\frac{\sigma^2\varepsilon^2u^2}{2n^2\alpha^2c^2}\right)^{\frac{n}{2n-1}}. \quad (56)$$

Hence

$$V_u=2\lambda\left(\frac{\varepsilon a_0^3}{c}\right)^{\frac{2}{3\lambda}}u^{\frac{2-3\lambda}{3\lambda}}-2\alpha n\left(\frac{\sigma^2\varepsilon^2}{2n^2\alpha^2c^2}\right)^{\frac{n}{2n-1}}u^{\frac{1}{2n-1}} \quad (58)$$

or

$$V_uu=2\lambda\left(\frac{\varepsilon a_0^3}{c}\right)^{\frac{2}{3\lambda}}u^{\frac{2}{3\lambda}}-2\alpha n\left(\frac{\sigma^2\varepsilon^2}{2n^2\alpha^2c^2}\right)^{\frac{n}{2n-1}}u^{\frac{2n}{2n-1}}, \quad (59)$$

So that we have

$$V_uu=\frac{2\lambda}{t^2}-2\alpha n\left(\frac{\sigma^2}{2n^2\alpha^2a_0^6t^{6\lambda}}\right)^{\frac{n}{2n-1}}=\varepsilon Y. \quad (60)$$

If

$$\lambda = \frac{2n-1}{3n}, \quad (61)$$

then we have

$$X = \left(\sqrt[2n-1]{\frac{\sigma^2}{2n^2\alpha^2 a_0^6}} \right) t^{-\frac{2}{n}}, \quad (62)$$

$$Y = 2\varepsilon^{-1}\left[\frac{2n-1}{3n} - \alpha n \left(\frac{\sigma^2}{2n^2\alpha^2 a_0^6} \right)^{\frac{n}{2n-1}} \right] t^{-2}, \quad (63)$$

$$V = (2n-1)\left[\frac{2n-1}{3n^2} - \alpha \left(\frac{\sigma^2}{2n^2\alpha^2 a_0^6} \right)^{\frac{n}{2n-1}} \right] t^{-2}, \quad (64)$$

$$K = \frac{2n-1}{3n^2} t^{-2}, \quad (65)$$

$$u = \frac{c}{\varepsilon a_0^3} t^{\frac{1-2n}{n}}. \quad (66)$$

In this case the potential has the form

$$V = (2n-1)\left[\frac{2n-1}{3n^2} - \alpha \left(\frac{\sigma^2}{2n^2\alpha^2 a_0^6} \right)^{\frac{n}{2n-1}} \right] \left(\frac{\varepsilon a_0^3 u}{c} \right)^{\frac{2n}{2n-1}}. \quad (67)$$

Finally, let us we present the expressions for the equation of state and deceleration parameters. For the our particular solution (63) they take the form

$$\omega_f = -1 + \frac{2n}{2n-1}, \quad q = \frac{n+1}{2n-1}. \quad (68)$$

These formulas tell us that for $n \in (1, 0{,}5)$ $[n \in (-\infty, -1)]$ and $n \in (0{,}5, \infty)$ we get the accelerated [decelerated] expansion phase of the universe.

In this paper, the g-essence model is studied. It is shown that this model admits some important physical reductions namely k-essence and f-essence. The exact solution of the g-essence model is constructed. This solution can describes the observed accelerated expansion of the universe.

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Резюме

Жұмыста g-эссенцияның моделі зерттеледі. Бұл модельде кейбір маңызды – k-эссенция және f-эссенция деп аталатын физикалық редукциялары көрсетілген. g-эссенция моделінің нақты шешімі тұрғызылған. Бұл шешімдер Әлемнің үдемелі кеңеюін түсіндіре алады.

Резюме

В работе исследуется модель g-эссенции. Показано, что эта модель содержит некоторые важные физические редукции, именуемые как k-эссенция и f-эссенция. Построены точные решения для модели g-эссенции. Эти решения могут описывать наблюдаемое ускоренное расширение Вселенной.

Eurasian National University,
Astana

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