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ON COMMUTATIVITY OF PO-MAXIMAL LATTICE ORDERED GROUPS

Annottation

A partially ordered structure (*M*, ≤, …) is called *po-maximal* if for any definable subset *A* and any maximal chain *B* in *M* there exist both the maximal and the minimal convex components of the intersection of *A* and *B*. In the paper we prove that any po-maximal lattice ordered group is abelian.

Since in 1986 Anand Pillay and Charles Cteinhorn [1–3] introduced and developed notion of o-minimality, investigations of non-stable, namely, of ordered structures became much more numerous and fruitful. Later on Dugald Macpherson, David Marker and Charles Cteinhorn developed weak o-minimality which is a generalization of o-minimality. Further generalization was pseudofinite o-minimality, introduced in [5]. KanatKudaibegrenov considered generalization of o-minimality to partial orderings in [6]. Here we suggest one more generalization of o-minimality on partially ordered structures.

Recall that a subset *A* of a partially ordered structure is called an *antichain*if for all a_1, a_2□A it holds that (a_1□ a_2□ a_1□ a_2). A subset *A* is *convex* if for all a, b □A it holds that □x (a<x<b → x□A). It is quite easy to see that anyantichain is convex, just because if the hypothesis of an implication is false, then this implication is true.

Also recall that a *chain* is a totally ordered subset of a poset and that a *convex component* of a subset B of a tottaly ordered set A is a maximal convex subset of B.

DefinitionA partially ordered structure (M, ≤, …) is called *po-maximal* if for any formula,any $\bar{a} \in M$ and any maximal chain *A* of *M* there exist both maximal and minimal convex component of$\varphi(M, \bar{a}) \cap A$.

Theorem 1Any definable subgroup of a po-minimal directed group G is convex.

Proof. Let *H*<*G* be definable. Assume the contrary, that *H* is not convex. Then there existh_1, h_2□*H*, *g*□*G* such that $h_1<g<h_2$ and *g*□*H*.

Note that $\{e < h_1^{-1}g < h_1^{-1}h_2\}$ and that $h_1^{-1}h_2 > e$.

Let A_1 be a chain which contains $h_1<g_2<h_2$and satisfies the condition that if an element *a* belongs to A_1 then both $ah_1^{-1}g$ and $ah_1^{-1}h_2$ belong to A_1. Let *A* be a maximal chain which contains A_1.

Note that

$$h_1 < g < h_1(h_1^{-1}h_2) < g(h_1^{-1}h_2) < h_1(h_1^{-1}h_2)^2 < g(h_1^{-1}h_2)^2 < \ldots$$

Let *B* be a maximal convex component of $A \cap H$. Let *b*□*B*, then $b < (h_1^{-1}g)b < b(h_1^{-1}h_2) < \ldots$

Since $h_1^{-1}h_2b$, $h_1^{-1}gb$, if *a*□*A* then $a(h_1^{-1}h_2) \in A$

By construction of the set *A*it holds that$bh_1^{-1}h_2 \in A$ .Since *B* is a maximal convex component of $A \cap H$, so $bh_1^{-1}h_2 \in B$. Then $bh_1^{-1}g \in B$ so $bh_1^{-1}g \in H$.

Recall that $bh_1 \in H \Rightarrow b^{-1}h_1 \in H \Rightarrow b^{-1}bh_1^{-1}g = h_1^{-1}g \in H$ .Then $h_1h_1^{-1}g = g \in H$ for a contradiction.

Recall notion of the centralizator: *C*(*a*)={*g*□*G*: *ga*=*ag*}, it is definable by the following formula $\varphi(x) = xa = ax$, $\varphi(G) = C(a)$.

Since*C*(*a*) is a definable subgroup, it is convex. If 1<*b*<*a*, then *ab*=*ba*, because 1, *a*□C(a) and $a^n \in C(a)$, because $a^n \cdot a = a^{n+1}$=*a* a^n.

In the book [7] by Blyths the following two theorems were proved.

Theorem 9.14*If G is a lattice-ordered group then, for all x, y* □*G,*

(3) $x = x_+ x_- = x_- x_+$;

(4) $x \le y$ *if and only if* $x_+ \le y_+$ *and* $x_- \le y_-$.

Let b≤a, then $1 \le b_+ \le a_+$ and $b_- \le a_- \le 1$

Theorem 9.16*For a lattice-ordered group G the following statements are equivalent* :

(1) *G is commutative*;

(2) P_G *is commutative*;

Theorem 2 Let *G*be a po-maximal lattice-ordered group. Then *G* is abelian.

Proof. Let *a, b*□*G*. Denote $b^+ = b \vee 1$ $a^- = a \wedge 1$ $b^- = b \wedge 1$

$a^+ = a \vee 1$

Let $P_G = \{g_+ : g$□$G\}$ where $g_+ = g \vee 1$. By Theorem 9.16 from [7] it is sufficient to prove that any two elements from the positive cone P_G commute.

So let $1 \le a$, $1 \le b$.Then $1 \le a \le ab$.

Since 1, *ab*□*C*(*ab*) and *C*(*ab*) is convex, so *a*□*C*(*ab*), that is *a*(*ab*)=(*ab*)*a*. By multiplying by a^{-1} from the left we obtain $a^{-1}a(ab) = a^{-1}(ab)a$, $ab = (a^{-1}a)ba = ba$. Thus $ab = ba$.

From now on we shall use the additive notation of the group operation.

Theorem 3 Let *G* be a po-maximal lattice-ordered group. Then *G* is divisible, that is for any positive integer *n* and for any element *g* of *G* the equation $nx = g$ has a solution in *G*.

Proof. Assume the contrary, that an element *g* is not *n*-divisible for some positive integer *n*.By theorem 9.14 of [7] it is sufficient to prove that any positive element is *n*-divisible. So we assume that the element *g* is positive.

Then in the chain

$g < ng < (n+1)g < 2ng < (2n+1)g < \dots$

elements with an odd place are not divisible by *n*, and the elements with an even place are divisible. So let a maximal chain *A* contains that sequence and is closed under adding *g*. Let *B* be the maximal convex component of the set of *n*-divisible elements from *A*. Let *b* be from the set *B*. Then $b + g$is not n-divisible and $b + ng$ is *n*-divisible. Note that $b < b + g < b + ng$. But since *B* is the maximal convex component, the element $b + ng$belongs to *B*, then by convexity $b + g$ belongs to *B* for a contradiction.

Theorem 4 Let *G* be a po-maximal lattice-ordered group. Then *G* is dense.

Proof. Assume the contrary that there is a minimal positive element *a*. But we know that the element *a* is 2-divisible, so there is an element *b* such that $2b = a$. It is well-known that then $b < a$ and $0 < b$, for a contradiction.

Theorem 5. Let *R* be a po-maximal lattice-ordered associative division ring. Then *K* is a field.

Proof. By Theorem 2 the multiplicative group of *R* is abelian, then it is a field.

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