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A NOTE ON 1D GENERALIZED TODA LATTICE

A generalized Toda Lattice equation is considered. The associated linear problem (Lax representation) is found. For the simple case $N = 3$ the τ-function Hirota form is presented that allows to construct an exast solutions of the equations of the 1DGTL. The corresponding hierarchy and its relations with the nonlinear Schrodinger equation and Heisenberg ferromagnetic equation are discussed.

1. Introduction. The 1D Toda lattice (1DTL)

$$\ddot{q}_n = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}}, \qquad (1.1)$$

is one of most important integrable equations in mathematics and physics. The equations (1.1) describe an interacting N particles, each with mass $m_n = 1$, arranged along a line at positions $q_1, q_2, \ldots, q_N$. Between each pair of adjacent particles, there is a force whose magnitude depends exponentially on the distance between them. The 1DTL was discovered by Morikasu Toda in 1967 [1]. Using the computer experiments, in [7] was suggested that the 1DTL is integrable. In [3–10] the integrability of the 1DTL is proved. Note that this equation is a discrete approximation of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \qquad (1.2)$$

There are exist several generalization of (1.1) [11–14]. In this paper we consider some integrable generalizations of 1DTL.

2. Basics of TL. In this section we present some very known fundamental information for the 1DTL. Let p_v denotes the momentum of the n-th particle. Then the total energy of the system is the Hamiltonian

$$H = \frac{1}{2}\sum_{n=1}^{N} p_n^2 + \sum_{n=1}^{N-1} e^{q_n - q_{n+1}}. \qquad (2.1)$$

So the system (1.1) can be written as

$$\dot{q}_n = \{H, q_n\} = \frac{\partial H}{\partial p_n},$$

$$p_n = \{H, p_n\} = -\frac{\partial H}{\partial q_n}. \qquad (2.2)$$

Except the original form (1.1), there are exist the various equivalent forms of the equation of TL (1.1). Some of them as follows.

i)

$$\dot{\alpha}_n = \alpha_n(\beta_n - \beta_{n+1}),$$

$$\dot{\beta}_n = \alpha_{n-1} - \alpha_n. \qquad (2.3)$$

ii)

$$\dot{a}_n = \alpha_n(b_{n+1} - b_n),$$

$$\dot{b}_n = 2(\alpha_n - \alpha_{n+1}). \qquad (2.4)$$

iii)

$$\left[D_t^2 - 4\sinh^2\left(\frac{D_n}{2}\right)\right] f_n \circ f_n = 0. \qquad (2.5)$$

iv)

$$\ddot{\tau}_n \tau_n - \dot{\tau}_n^2 = \tau_{n+1}\tau_{n-1} = 0 \qquad (2.6)$$

and so on. Above D_t, D_v are the well known Hirota bilinear operators and τ_n is so-called τ-function which play a key role of the theory of integrable systems. Note that a new and initial "physical" (q_n, p_n) dependent variables are related as

$$\alpha_n = e^{q_n - q_{n-1}}, \quad \beta_n = p_n,$$

$$a_n = \frac{1}{2} e^{\frac{1}{2}(q_n - q_{n-1})}, \quad b_n = -\frac{1}{2} p_n,$$

$$\ddot{\ln} f_n = (\ln f_n)_{tt} = e^{q_{n-1} - q_n}, \qquad (2.7)$$

$$q_n = \ln\frac{\tau_{n+1}}{\tau_n}.$$

There are at least two possible Lax matrices for the TL, one of order 2×2, another one of order N×N. The 2×2 Lax pair is defined as

$$L = \begin{pmatrix} p_n + \lambda & -e^{q_n} \\ -e^{-q_n} & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -e^{q_n} \\ -e^{q_n} & \lambda \end{pmatrix}. \qquad (2.8)$$

The associated linear problem is

$$\Psi_{n+1} = L_n \Psi_n,$$

$$\Psi_{nt} = M_n \Psi_n. \qquad (2.9)$$

The compatibility condition of these equation is

$$L_{nt} + L_n M_n - M_{n+1} L_n = 0. \qquad (2.10)$$

Here

$$L_n = \sum_{1}^{N} L^v X_v, \quad M_n = \sum_{v=1}^{N} M^v X_v, \qquad (2.11)$$

where $\left[X_{\mu,}X_{\nu}\right]=C^{\lambda}_{\mu\nu}X_{\lambda}$. Then the TL equations obtained from the Lax equation

$$\dot{L}=[L,M]. \qquad (2.12)$$

There are exist a so-called r-matrix representation for the Poisson brackets $\{L^{\mu},L^{\nu}\}$ between the matrix elements L. It has the form

$$\{L\otimes,L\}=[r,L\otimes I+I\otimes L]=[r,L_1+L_2], \qquad (2.13)$$

or

$$\{L\otimes,L\}=$$
$$=[r,L\otimes I]-[r^{T},I\otimes L]=[r,L_1+L_2], \qquad (2.14)$$

where

$$r=\sum_{\mu,\nu=1}^{N}r^{\mu\nu}X_{\mu}\otimes X_{\nu},\ r^{T}=\sum_{\mu,\nu=1}^{N}r^{\nu\mu}X_{\mu}\otimes X_{\nu},\ L_1=$$
$$=L\otimes I,\ L_2=I\otimes L. \qquad (2.15)$$

Finally we have

$$\{L\otimes,L\}=\left(r^{\nu\tau}C^{\mu}_{\tau\lambda}L^{\lambda}-r^{\mu\tau}C^{\mu}_{\tau\lambda}L^{\lambda}\right)X_{\mu}\otimes X_{\nu}. \qquad (2.16)$$

In this notes we use the following form of the r-matrix

$$r=\sum_{i=1}^{N}E_{ii}\otimes E_{ii}+2\sum_{i,j=1(i<j)}^{N}E_{ij}\otimes E_{ji}. \qquad (2.17)$$

3. Generalized TL. In this paper we consider the following GTL

$$\dot{p}_k=2\left(a_{k-1}b_{k-1}-a_kb_k\right)+2u\upsilon\left(\delta_{k1}-\delta_{k-1}\right),$$
$$\dot{a}_k=\left(p_k-p_{k+1}\right)a_k+2\upsilon\left(b_{k-1}\delta_{k,0}-b_{k+1}\delta_{k-1}\right),$$
$$\dot{b}_k=\left(p_k-p_{k+1}\right)b_k+2u\left(a_{k-1}\delta_{k,0}-a_{k+1}\delta_{k,-1}\right),$$
$$\dot{u}=\left(p_2-p_4\right)u,$$
$$\dot{\upsilon}=\left(p_2-p_4\right)\upsilon. \qquad (3.1)$$

It can be written in Lax form as (2.2) with

$$L=\begin{pmatrix} p_{-N} & a_{-N} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\ b_{-N} & \ddots & \ddots & \ddots & & & & & \vdots\\ 0 & \ddots & \ddots & \ddots & 0 & & & & \vdots\\ \vdots & \ddots & \ddots & p_{-1} & a_{-1} & \upsilon & & & \vdots\\ \vdots & & 0 & b_{-1} & p_0 & a_0 & 0 & & \vdots\\ \vdots & & & u & b_0 & p_1 & \ddots & \ddots & \vdots\\ \vdots & & & & 0 & \ddots & \ddots & \ddots & 0\\ \vdots & & & & & \ddots & \ddots & \ddots & a_{N-1}\\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & b_{N-1} & p_N \end{pmatrix}, \qquad (3.2)$$

$$M=\begin{pmatrix} 0 & a_{-N} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\ -b_{-N} & \ddots & \ddots & \ddots & & & & & \vdots\\ 0 & \ddots & \ddots & \ddots & 0 & & & & \vdots\\ \vdots & \ddots & \ddots & 0 & a_{-1} & \upsilon & & & \vdots\\ \vdots & & 0 & -b_{-1} & 0 & a_0 & 0 & & \vdots\\ \vdots & & & -u & -b_0 & 0 & \ddots & \ddots & \vdots\\ \vdots & & & & 0 & \ddots & \ddots & \ddots & 0\\ \vdots & & & & & \ddots & \ddots & \ddots & a_{N-1}\\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -b_{N-1} & 0 \end{pmatrix}. \qquad (3.3)$$

There are take place a Lie-Poisson brackets [2]

$$\{p_i,a_i\}=a_i,$$
$$\{p_{i+1},a_i\}=-a_i,$$
$$\{p_i,b_i\}=b_i,$$
$$\{p_{i+1},b_i\}=-b_i,$$
$$\{p_{-1},u\}_M=u,$$
$$\{p_1,u\}_M=-u,$$
$$\{p_{-1}v\}_M=\upsilon, \qquad (3.4)$$
$$\{p_1,v\}_M=-\upsilon,$$
$$\{a_{-1},a_0\}_M=2\upsilon,$$
$$\{b_{-1},b_0\}_M=2u.$$

All other brackets are zero. We denote this bracket by π_1. The functions

$$H_i=\frac{1}{i}\mathrm{tr}L^{i} \qquad (3.5)$$

are independent invariants in involution that is

$$\{H_i,H_j\}=0. \qquad (3.6)$$

The expressions for H_j are, for example,

$$H_1 = \sum_{k=-N}^{N} p_k,$$

$$H_2 = \frac{1}{2}\sum_{k=-N}^{N} p_k^2 + \sum_{k=-(N-1)}^{-1} a_k b_k + u\upsilon,$$

$$H_3 = \frac{1}{3}\sum_{k=-N}^{N} p_k^3 +$$

$$+\sum_{k=1}^{N}\left[a_1 b_1 p_1 + \left(a_1 b_1 + a_2 b_2 + u\upsilon\right)p_2 +\right.$$

$$+\left(a_2 b_2 + a_3 b_3\right)p_3 + \left(a_3 b_3 + a_4 b_4 + u\upsilon\right)p_4 +$$

$$\left. + a_4 b_4 p_5 + a_2 a_3 u + b_2 b_3 \upsilon\right]$$

and so on. The invariant H_1 is the only Casimir. The Hamiltonian in this bracket is $H_2 = \frac{1}{2}\mathrm{tr}L^2$. As for the usual TL in our case we can introduce the quadratic Toda brackets which appears in conjunction with isospectral deformations of Jacobi matrices. It is a Poisson bracket in which the Hamiltonian vector field generated by H_1 is the same as the Hamiltonian vector field generated by H_2 with respect to the π_1 bracket. We will denote this Poisson bracket by π_2. The bracket π_2 is easily defined by taking the Lie derivative of π_1 in the direction of suitable master symmetry. This bracket has det(L) as Casimir and H_2 = tr$L$ is the Hamiltonian. The eigenvalues of L are still in involution. Furthermore, π_2 is compatible with π_1. We also have

$$\pi_2 dH_i = \pi_1 dH_{i+1}. \tag{3.8}$$

Note that both brackets π_1 and π_2 transforms to the corresponding brackets of the usual TL for the case $u = \upsilon = 0$.

4. Case $N = 3$. Let $N = 3$. Then the equations of the GTL take the form

$$\begin{aligned}
\dot p_1 &= -2\left(a_1^2 + u^2\right),\\
\dot p_2 &= 2\left(a_2^2 - a_1^2\right),\\
\dot p_3 &= 2\left(a_2^2 + u^2\right),\\
\dot a_1 &= a_1\left(p_1 - p_2\right) - 2ua_2,\\
a_2 &= a_2\left(p_2 - p_2\right) + 2ua_1,\\
\dot u &= \left(p_1 - p_3\right)u.
\end{aligned} \tag{4.1}$$

This system can be written in Lax form as

$$\dot L = [L, M], \tag{4.2}$$

where

$$L = \begin{pmatrix} p_1 & a_1 & u \\ a_1 & p_2 & a_2 \\ u & a_2 & p_3 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & a_1 & u \\ -a_1 & 0 & a_2 \\ -u & -a_2 & 0 \end{pmatrix}. \tag{4.3}$$

There exists a Lie-Poisson bracket given by the formula

$$\begin{aligned}
\{p_i, a_i\} &= a_i,\\
\{p_{i+1}, a_i\} &= -a_i,\\
\{p_1 u\} &= u,\\
\{p_3 u\} &= -u,\\
\{a_1 a_2\} &= 2u.
\end{aligned} \tag{4.4}$$

All other brackets are zero. We denote this bracket by π_1. The functions

$$H_i = \frac{1}{i}\mathrm{tr}L^i \tag{4.5}$$

are independent invariants in involution that is

$$\{H_i, H_j\} = 0.$$

The expressions for H_i are, for example,

$$H_1 = \sum_{k=1}^{3} p_k,$$

$$H_2 = \frac{1}{2}\sum_{k=1}^{3} p_k^2 + \sum_{k=1}^{2} a_k^2 + u^2,$$

$$H_3 = \frac{1}{2}\sum_{k=1}^{3} p_k^3 +$$

$$+\sum_{k=1}^{3}\left[a_1^2 p_1 + \left(a_1^2 + a_2^2 + u^2\right)p_2 + \left(a_2^2 + a_3^2\right)p_3 +\right.$$

$$\left. + \left(a_3^2 + a_4^2 + u^2\right)p_4 + a_4^2 p_5 + 2a_2 a_3 u\right] \tag{4.6}$$

and so on. The Hamiltonian in this bracket is $H_2 = \frac{1}{2}\mathrm{tr}L^2$. The Casimirs of the system are

$$\begin{aligned}
C_1 &= H_1 = p_1 + p_2 + p_3,\\
C_2 &= \frac{a_1 a_2}{u} = -2p_2,\\
C_3 &= \frac{\upsilon}{u_1} = d_3.
\end{aligned} \tag{4.7}$$

Note that

$$\dot{C}_k = 0.$$

4.1. 1GTL q_k, p_k coordinates. Let us write the system (4.1) in terms of the coordinates q_k, $p_k = \dot{q}_k$, $k = 1,2,3$. Then from (4.1) we have

$$\begin{aligned} \ddot{q}_1 &= -2(a_1^2 - u^2), \\ \ddot{q}_2 &= -2(a_2^2 - u^2), \\ \ddot{q}_3 &= 2(a_2^2 - u^2), \\ \dot{a}_1 &= a_1 = (p_1 - p_2) - 2ua_1, \\ \dot{u}_2 &= p_{13}u. \end{aligned} \qquad (4.8)$$

Hence we get

$$u = u_0 + e^{q_{12}},\ q_{ij} = q_i - q_j,\ p_{ij} = p_i - p_j,\quad u_0 = \text{const}. \qquad (4.9)$$

Let $a_1 = e^{q_{12}} p_4$, $a_2 = e^{q_{23}} p_4$ and

$$L = \begin{pmatrix} p_1 & e^{q_{12}} p_4 & e^{q_{13}} \\ d_1 e^{q_{12}} p_5 & p_2 & e^{q_{23}} q_4 \\ d_1 d_2 e^{q_{23}} & d_2 e^{q_{23}} q_5 & p_3 \end{pmatrix}. \qquad (4.10)$$

Then the equations of motion become

$$\begin{aligned} \ddot{q}_1 &= -2(a_1^2 - u^2), \\ \ddot{q}_2 &= -2(a_2^2 - u^2), \\ \ddot{q}_3 &= 2(a_2^2 - u^2), \\ \dot{p}_4 &= 2e^{q_{13}+q_{23}+q_{12}} q_4 = -2e^{2q_{12}} q_4, \\ \dot{q}_4 &= 2e^{q_{13}+q_{23}+q_{12}} p_4 = -2e^{2q_{13}} p_4, \\ \dot{u}_2 &= p_{13}u. \end{aligned} \qquad (4.11)$$

So we see that $p_4 = \frac{1}{2}e^{2q_{31}}\dot{q}_4 \neq \dot{q}_4$ and

$$\begin{aligned} \ddot{q}_1 &= -2(e^{2q_{12}} p_4 + e^{2q_{13}}), \\ \ddot{q}_2 &= -2(q_4^2 e^{2q_{23}} - p_4^2 e^{2q_{12}}), \\ \ddot{q}_3 &= -2(e^{2q_{23}} p_4^2 + e^{2q_{13}}), \\ \ddot{q}_4 &= 4(p_{12} p_4 e^{2q_{23}} - q_4 e^{2q_{12}}), \\ \dot{u} &= p_{13}u \end{aligned} \qquad (4.12)$$

or

$$\begin{aligned} \ddot{q}_1 &= -2(e^{2q_{12}} p_4 + e^{2q_{13}}), \\ \ddot{q}_2 &= -2(q_4^2 e^{2q_{23}} - p_4^2 e^{2q_{12}}), \\ \ddot{q}_4 &= 4(p_{12} p_4 e^{2q_{23}} - q_4 e^{2q_{12}}). \end{aligned} \qquad (4.13)$$

Here

$$C_4 = p_4 q_4 + ae^{q_4 q_4}. \qquad (4.14)$$

4.2. In P_i, Q_i coordinates. Let us consider a new representation for the Lax matrix L as

$$L = \begin{pmatrix} P_1 & e^{Q_1} P_3 & e^{Q_1+Q_2} \\ d_1 e^{Q_1} P_4 & P_2 - P_1 & e^{Q_2} Q_3 \\ d_1 d_2 e^{Q_1+Q_2} & d_2 e^{Q_2} Q_4 & -P_2 \end{pmatrix}, \qquad (4.15)$$

where $\{P_i, Q_j\} = \delta_{ij}$. In this case the equations of motion take the form

$$\begin{aligned} \dot{P}_1 &= Q_1, \\ \dot{P}_2 &= Q_2, \\ \dot{P}_3 &= Q_3, \\ \dot{Q}_1 &= a_1(p_1 - p_2) - 2ua_2, \\ \dot{Q}_2 &= a_2(p_2 - p_3) - 2ua_1, \\ \dot{Q}_3 &= a_2(p_2 - p_3) - 2ua_1, \\ \dot{u} &= (P_1 - P_3)u. \end{aligned} \qquad (4.16)$$

4.3. Solutions. To find solutions we first introduce a new variables as

$$c_k(t) = p_k\left(\frac{1}{2}t\right),\quad d_{k+1}(t) = a_k^2\left(\frac{1}{2}t\right),\quad \omega = u^2\left(\frac{1}{2}t\right). \qquad (4.17)$$

Then the system (4.1) becomes

$$\begin{aligned} \dot{c}_1 &= -(d_2 + \omega), \\ \dot{c}_2 &= d_2 - d_3, \\ \dot{c}_3 &= d_3 + \omega, \\ \dot{d}_2 &= c_{12} d_2 - 2\sqrt{\omega d_3}, \\ \dot{d}_3 &= c_{23} d_3 + 2\sqrt{\omega d_2}, \\ \dot{\omega} &= c_{13}\omega. \end{aligned} \qquad (4.18)$$

The GTL equation (4.18) is the isospectral $(\lambda_t = 0)$ and obeys the compatibility condition of the following spectral problems:

$$\dot{\phi}_n = c_{n-1}\phi_n + \phi_{n-1},$$
$$d_n\phi_{n+1} + c_{n-1}\phi_n + \phi_{n-1} = \lambda\phi_n, \qquad (4.19)$$

where λ is a spectral parameter. Namely, it has the Lax representation $L_t = [M,L]$, where the Lax pair is defined by

$$L_{nm} = d_n\delta_{n+1,m} + c_{n-1}\delta_{nm} + \delta_{n-1,m},$$
$$M_{nm} = \delta_{n+1,m} + c_{n-1}\delta_{nm}. \qquad (4.20)$$

On the other hand the matrix form these matrices have the form

$$L = \begin{pmatrix} c_0 & d_1 & \omega \\ 1 & c_1 & d_2 \\ 0 & 1 & c_2 \end{pmatrix}, \quad M = \begin{pmatrix} c_0 & 1 & 0 \\ 0 & c_1 & 1 \\ 0 & 0 & c_2 \end{pmatrix}. \qquad (4.21)$$

Using an dependent variable transformation

$$d_k = 1 + (\ln\tau_k)_{tt} = \left(\ln\tau_k e^{\frac{1}{2}t^2}\right)_{tt},$$

$$\omega = -1 - (\ln f)_{tt} = -\left(\ln fe^{\frac{1}{2}t^2}\right)_{tt}, \qquad (4.22)$$

we find

$$c_1 = I_1 - \left(\ln\frac{\tau_2}{f}\right)_t,$$

$$c_2 = I_2 + \left(\ln\frac{\tau_2}{\tau_3}\right)_t, \qquad (4.23)$$

$$c_3 = I_3 + \left(\ln\frac{\tau_3}{f}\right)_t.$$

Hence we get

$$c_{12} = I_{12} - \left(\ln\frac{\tau_2^2}{\tau^3 f}\right)_t,$$

$$c_{23} = I_{23} + \left(\ln\frac{\tau_3 f}{\tau_3^2}\right)_t, \qquad (4.24)$$

$$c_{13} = I_{13} - \left(\ln\frac{\tau_2\tau_3}{f^2}\right)_t.$$

To define the unknown functions we have the system

$$(d_2 + d_3)_t = c_{12}d_2 + c_{23}d_3,$$
$$(\dot{d}_3 + c_{23}d_3)^2 = 4\omega d_2, \qquad (4.25)$$
$$\dot{\omega} - c_{13}\omega = 0.$$

Hence and from we get

$$(\ln\tau_2\tau_3)_{ttt} = \left[I_{12} - \left(\ln\frac{\tau_2^2}{\tau_3 f}\right)_t\right]\left(\ln\tau_2 e^{\frac{1}{2}t^2}\right)_{tt} -$$

$$-\left[I_{23} + \left(\ln\frac{\tau_2 f}{\tau_3^2}\right)_t\right]\left(\ln\tau_3 e^{\frac{1}{2}t^2}\right)_{tt} = 0,$$

$$\left\{(\ln\tau_3)_{ttt} - \left[I_{23} - \left(\ln\frac{\tau_2 f}{\tau_3^2}\right)_t\right]\left(\ln e^{\frac{1}{2}t^2}\right)_{tt}\right\}^2 +$$

$$+4\left(\ln\tau_2 e^{\frac{1}{2}t^2}\right)_{tt}\left(\ln fe^{\frac{1}{2}t^2}\right)_{tt} = 0,$$

$$\ddot{f}f - \dot{f}^2 + f^2 + \frac{f^4}{\tau_2\tau_3}e^{I_{13t}} - I_{13t} = 0. \qquad (4.26)$$

For simplicity we set $I_{ij} = 0$. Then the system (4.26) becomes

$$(\ln\tau_2\tau_3)_{ttt} + \left(\ln\frac{\tau_2^2}{\tau_3 f}\right)_t\left(\ln\tau_2 e^{\frac{1}{2}t^2}\right)_{tt} -$$

$$-\left(\ln\frac{\tau_2 f}{\tau_3^2}\right)_t\left(\ln\tau_3 e^{\frac{1}{2}t^2}\right)_{tt} = 0,$$

$$\left[(\ln\tau_3)_{ttt} - \left(\ln\frac{\tau_2 f}{\tau_3^2}\right)_t\left(\ln e^{\frac{1}{2}t^2}\right)_{tt}\right]^2 +$$

$$+4\left(\ln\tau_k e^{\frac{1}{2}t^2}\right)_t\left(\ln fe^{\frac{1}{2}t^2}\right)_{tt} = 0,$$

$$\ddot{f}f - \dot{f}^2 + f^2 + \frac{f^4}{\tau_2\tau_3}e^{I_{13t}} + I_{13t} = 0. \qquad (4.27)$$

We expand the functions τ_n, f in a formal power series in an arbitrary parameter ε as

$$\tau_n = \sum_{k=0} \varepsilon^k \tau_n^{(k)} = 1 + \varepsilon\tau_n^{(1)} + \varepsilon^2\tau_n^{(2)} +, \ldots,$$

$$f = \sum_{k=0} \varepsilon^k f^{(k)} = 1 + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} +, \ldots. \quad (4.28)$$

Expanding the l.h.s. of (4.27) in ε and equating corresponding coefficients, the resulting equations can be written in the form.

5. The 1D GTL hierarchy. First let us recall the main formulas of the usual TL hierarchy. The corresponding hierarchy is defined by

$$\frac{\partial L}{\partial t_k} = [L, B_k], \left(B_k = (L^k)_-,\ k = 1, 2, 3 \ldots\right). \quad (5.1)$$

The τ-functions of the TL hierarchy obey the following equations

$$[D_k - h_k(D)]\tau_{n+1} \cdot \tau_n = 0,\ \left(k = 1, 2, 3 \ldots\right). \quad (5.2)$$

Here

$$e^{\sum_{k=1}^{\infty} \frac{1}{k} D_k z^k} = \sum_{n=0}^{\infty} h_n(D) \varepsilon^n,$$

$$D = (D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \frac{1}{4}D_4, \cdots). \quad (5.3)$$

It is interesting to note that the nonlinear Schrodinger equation (NLSE) is the second member of the TL hierarchy. In fact from (5.2) as $k = 2$ and from (2.6) we get the following set of equations (for example [2])

$$(D_2 - D_1^2)\tau_{n+1} \cdot \tau_n = 0,$$
$$D_1^2 \tau_n \cdot \tau_n = 2\tau_{n+1} \cdot \tau_{n-1}. \quad (5.4)$$

This set is equivalent to the NLSE

$$i\phi_{t_2} + \phi_{t_1 t_1} + 2\phi^2\bar{\phi} = 0, \quad (5.5)$$

where

$$\phi = \tau_{n+1}\tau_n^{-1},\ \bar{\phi} = \tau_{n-1}\tau_n^{-1},\ t_2 \to it_2. \quad (5.6)$$

The set (5.4) is the compatibility condition of the following set of linear equations

$$\psi_{t_1} = \begin{pmatrix} 0 & \tau_{n-1}\tau_n^{-1} \\ \tau_{n+1}\tau_n^{-1} & 0 \end{pmatrix}\psi,$$

$$\psi_{t_2} = i\begin{pmatrix} \tau_{n-1}\tau_{n+1}\tau_n^{-2} & -(\tau_{n-1}\tau_n^{-1})_{t_1} \\ (\tau_{n-1}\tau_n^{-1})_{t_1} & -\tau_{n-1}\tau_{n+1}\tau_n^{-2} \end{pmatrix}\psi. \quad (5.7)$$

Note that in this case the matrix $S = \psi^{-1}\sigma_3\psi$ obeys the Heisenberg ferromagnetic equation

$$2iS_{t_2} = [S, S_{t_1 t_1}]. \quad (5.8)$$

Finally note that the 1DGTL hierarchy has the same form as (5.1) but content the additional two equations related with u and υ.

6. Conclusion. In the present Letter we considered one of integrable generalizations of 1DTL. The corresponding Lax representation is presented. For the particular case $N = 3$ the bilinear form (τ-function form) is found that allows to construct exast solutions of the studied generalized Toda equation.

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Резюме

Тодтың жалпыланған торы талқыланады. (Лакс ұсынысы) сызықты есептің шешімі табылады. $N = 3$ болған жағдайда, Хиротаның τ-функциясы ұсынылып, Тодтың жалпыланған теңдеуіндегі 1D нақты шешімдері тұрғызылды. Ферромагниттер үшін Шредингердің сызықты емес тендеуімен Гейзенберг тендеуі арасындағы байланыс және сәйкес келетін иерархия талқыланды.

Резюме

Рассматривается обобщенная решетка Тоды. Найдено решение линейной задачи (в представлении Лакса). Для случая $N = 3$ представлена τ-функция Хироты, позволяющая построить точные решения 1D обобщенного уравнения Тоды. Обсуждается соответствующая иерархия, связь с нелинейным уравнением Шредингера и с уравнением Гейзенберга для ферромагнетиков.

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