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STIRLING NUMBERS OVER RESTRICTED PARTITIONS OF SETS

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We introduce the properties of generalized Stirling numbers of the second kind which count the number of restricted set partitions.

1. Introduction. Suppose that $k_1, \dots, k_n$ are positive integers and $K = k_1 + \dots + k_n$.

Let $S((k_1, \dots, k_n), m)$ be the number of partitions of the set $\{1, \dots, K\}$ into m blocks such that

- the first k_1 elements $(1, \dots, k_1)$ must be in distinct blocks;
- the next k_2 elements $(k_1 + 1, \dots, k_1 + k_2)$ must be in distinct blocks
- and so on;
- the last k_n elements $(k_1 + \dots + k_{n-1} + 1, \dots, k_1 + \dots + k_n)$ must be in distinct blocks.

Example. $S((2,2,1),2) = 4$; the pairs of elements 1, 2 and 3, 4 cannot be in one block and thus the only four corresponding partitions are:

$$\{\{1, 3\}, \{2, 4, 5\}\}; \{\{1, 3, 5\}, \{2, 4\}\}; \{\{1, 4\}, \{2, 3, 5\}\}; \{\{1, 4, 5\},\{2, 3\}\}.$$

The number $S((k_1, \dots, k_n), m)$ can be considered as a generalization of the Stirling number of the second kind because $S((1, \dots, 1), m) = S(n, m)$, where $S(n, m)$ is a usual Stirling number of the second kind.

The case $S((r, 1,\dots,1), m)$ generalizes the r-Stirling number of the second kind introduced by Broder [1].

2. Properties. Note that the value $S((k_1, \dots, k_n), m)$ will remain the same if we arbitrarily permute the numbers $(k_1, \dots, k_n)$.

Now let us introduce the general formula for $S((k_1, \dots, k_n), m)$.

Let $a_{(j)} = a(a-1)\dots(a-j+1)$ be a falling factorial.

Theorem 1. There holds the general formula

$$S((k_1,\dots,k_n),m) = \frac{1}{m!}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i} i_{(k_1)}\dots i_{(k_n)}.$$

Proof. Let $A((k_1,\dots,k_n),m)$ be the number of surjective functions $f:\{1,\dots,K\}\to\{1,\dots,m\}$ for which $f(1),\dots,f(k_1)$ have distinct values; $f(k_1+1),\dots,f(k_1+k_2)$ have distinct values; and so on; $f(k_1+\dots+k_{n-1}+1),\dots,f(k_1+\dots+k_n)$ have distinct values.

Then $A((k_1,\dots,k_n),m)$ can be computed using the inclusion-exclusion principle:

$$A((k_1,\dots,k_n),m) = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i} i_{(k_1)}\dots i_{(k_n)}.$$

And it is also clear that

$$S((k_1,\dots,k_n),m) = \frac{1}{m!}A((k_1,\dots,k_n),m)$$

The recurrence relations are shown next.

Theorem 2. The following properties are true:

if $n = 1$, then $S((k),k) = 1$ and $S((k),i) = 0$ if $i \neq k$;

and the recurrence relations

$$S((k_1,...,k_n),m) = S((k_1,...,k_n-1),m-1) + (m-k_n+1)S((k_1,...,k_n-1),m); \qquad (1)$$

and

$$S((k_1,...,k_n,k_{n+1}),m) = \sum_{j=0}^{k_{n+1}} \binom{k_{n+1}}{j} (m+j-k_{n+})_{(j)} S((k_1,...,k_n),m+j-k_{n+1}). \qquad (2)$$

Proof. Equation (1). Consider the last element *K*. It can form one separate block and this provides $S((k_1,...,k_n-1),m-1)$ ways; or the last element is contained in some block with other elements and this gives $(m-k_n+1)S((k_1,...,k_n-1),m)$ because there are only $(m-k_n+1)$ proper blocks which do not contain the elements of $\{k_1+...+k_{n-1}+1,...,k_1+...+k_n-1\}$.

Equation (2). Consider the group of last k_{n+1} elements. Note that for each any $j = 0,...,k_{n+1}$ exactly *j* elements can share common blocks with other elements and thus, other $(k_{n+1}-j)$ elements form $(k_{n+1}-j)$ separate blocks of one element each.

This argument clearly gives the needed.

3. Connections with composition of differential operators. Let *D* be the derivative operator *d*/*dx*. For any fixed positive integer *k* let us consider the operator $E_k = x^k D^k$.

If $k = 1$, then $E = E_1 = xD$ has remarkable properties. For instance, it is well known that

$$E^n = \sum_{m=1}^{n} S(n,m) x^m D^m = \sum_{m=1}^{n} S(n,m) E_m,$$

where $S(n, m)$ is a Stirling number of the second kind.

Theorem 3.

$$E_{k_1} ... E_{k_n} = \sum_{i=0}^{K} S((k_1,...,k_n),i) x^i D^i = \sum_{i=0}^{K} S((k_1,...,k_n),i) E_i,$$

Proof. By induction on *n* and directly using the recurrence relation (2).

Note that such property was established in the problem of boson normal ordering [2], where numbers $S((k_1,...,k_n),m)$ were considered with another (less natural) combinatorial interpretation. Authors also provide several properties which generalize properties of Stirling numbers of the second kind, e.g., the polynomial identity (Cor. 4.1)

$$\prod_{i=1}^{n} x_{(k_i)} = \sum_{i=0}^{K} S((k_1,...,k_n),i) x_{(i)}.$$

REFERENCES

1. Broder A.Z. The *r*-Stirling numbers, *Discrete Math.* **1984**, 49, 241-259.

2. *Mendez M.A., Blasiak P., Penson K.A.* Combinatorial approach to generalized Bell and Stirling numbers and boson normal ordering problem, *J. Math. Phys.* **2005**, 46, 083511-1-8.

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ЕКІНШІ ТЕКТІ СТИРЛИНГ САНЫН ТАЛДАП ҚОРЫТУ

Мақалада шектеулі көптіктің бөлінуін есептеу арқылы анықталатын екінші текті Стирлинг санын талдап қорыту зерттелген.

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ОБОБЩЕННЫЕ ЧИСЛА СТИРЛИНГА ВТОРОГО РОДА

В статье изучаются обобщенные числа Стирлинга второго рода, которые определяются при счете разбиений множеств с ограничениями.