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ABOUT EXACT SOLUTION OF THE NONLINEAR EQUATION

Soliton solutions are the important aspects of multiplies field theories as classic, as quantum physics. Notion of the integrability allows making useful classification of the soliton systems. Soliton solutions of these systems have simple behavior in bumping and they are stable. By using such methods as the inverse problem scattering method we can construct multisoliton solutions of the nonlinear equations in the exact form.

There are some examples of the nonlinear equations of the integrable systems in (2+1)-dimensions, majority of them are no relativistic [1–3].

It was shown, that many well-known integrable equations in (1+1)-dimensions are reductions of the selfdual Yang-Mills-Higgs equation. Similarly, integrable systems in (2+1)-dimension we can obtain from the self dual Yang-Mills-Higgs equation as it's reductions. We note, that there are some examples of the multiplier solitons of the nonlinear systems as skirmions, monopoles, lamps. Such theories are relativistic and solitons have the topological nature [4, 5]. Self dual Yang-Mills-Higgs equation is the important example of the integrable system. Researching of such integrable model in (2+1)-dimensions demonstrated that soliton's dynamics can be nontrivial on the integrable models.

Yang-Mills-Higgs equation were widely investigated and the equation are known to integrable [6]. The Lax pair of this equation was obtained by R.S.Ward [6]. In the present paper, we consider the Yang-Mills-Higgs equation in the (2+1)-dimensional anti de Sitter space-time. There are two curved space-time with constant curvature: de Sitter space with positive scalar curvature; and anti de Sitter space with negative scalar curvature. At the same time, this equation is a reduction of the Gauss-Mainardi-Codazzi equation in (2+1)-dimensional anti de Sitter space-time. Here we find the soliton solution of the YMH equation in (2+1)-dimensional anti de Sitter space-time for the simplest non-Abelian group *SU(2)*. Let *M* be a three dimensional Lorenz manifold with metric g, A_μ is a gauge potential, L is a scalar Higgs field, both of which are valued in the Lie algebra of Lie group G. We suppose G is a matrix Lie group G. We denote the domain Ω: { the (2+1)-dimensional anti de Sitter space-time is the universal covering space of the hyperboloid $U^2+V^2-X^2-Y^2=1$ in $R^{2,2}$ with the metric $ds^2=-dU^2-dV^2+-dX^2+-dY^2$ }.

We define

$$r=\frac{1}{U+X},\quad x=\frac{Y}{U+X},\quad t=\frac{V}{U+X},\tag{1}$$

then a part of the 2+1 dimensional anti de Sitter space-time with $U+X>0$ is represented by the Poincare coordinates *(r,x,t)* with *r>0* and the metric is

$$ds^2=r^{-2}(-dt^2+dr^2+dx^2)=r^{-2}(dt^2+dudv),\tag{2}$$

where *u=x+t, v=x-t*. The Yang-Mills-Higgs field in the domain Ω satisfies the Yang-Mills-Higgs equation [5]

$$D\Phi=*F.\tag{3}$$

Or, written in terms of the components

$$D_\mu\Phi=\frac{1}{2\sqrt{|g|}}g_{\mu\nu}\varepsilon^{\nu\alpha\beta}F_{\alpha\beta},\tag{4}$$

where $D_\mu\Phi=\partial_\mu\Phi+[A_\mu,\Phi]$, $\partial_\mu=\frac{\partial}{\partial x^{\mu F}}$, $\{F_{\mu\nu}\}$ is the curvature corresponding to $\{A_\mu\}$, $F_{\mu\nu}=[D_\mu,D_\nu]$. The equation (4) is an integrable system in the sense that a Lax pair exists, if the metric g has constant curvature $R=$ const. In our case the curvature is negative $R<0$. With the Poincare coordinates (1), the equation (4) becomes

$$D_x\Phi=rF_{ur},\quad D_v\Phi=-rF_{vr},\quad D_r\Phi=-2rF_{uv}.\tag{5}$$

The system of nonlinear partial differential equations has a Lax pair

$$\begin{gathered}(rD_r+\Phi-2(\xi-u)D_u)\psi=0,\\ (2D_v+\frac{\xi-u}{r}D_r-\frac{\xi-u}{r^2}\Phi)\psi=0,\end{gathered}\tag{6}$$

where $D_\mu\psi=\partial_\mu\psi+A_\mu\psi$ and ξ is a complex spectral parameter. Thus (5) is the integrability condition of the over-determined system (6). We formulate the following statement. If the equation (5) has a trivial solution in the domain Ω, so the equation (5) has onesoliton solution with constant spectral parameter in the domain Ω of the following form

$$\overline{A}_u = 0,\ \overline{A}_v = -(\partial_r S)S^{-1} = \frac{1}{2r}(\partial_r S) - \frac{1}{2r^2}[\kappa\sigma_3, S]; \tag{7.1}$$

$$\overline{A}_r = -\frac{1}{2}(\partial_r S)S^{-1} + \frac{1}{2r}(S\sigma_3 S^{-1} - \sigma_3) = -\frac{1}{2r}(\partial_u S + 1); \tag{7.2}$$

$$\widetilde{\Phi} = -\frac{r}{2}(\partial_r S)S^{-1} + \frac{1}{2}(S\sigma_3 S^{-1} + \sigma_3) = -\partial_u S - 1 + \kappa\sigma_3, \tag{7.3}$$

where

$$S = \frac{\xi_0 - \overline{\xi}_0}{1+|\sigma|^2}\begin{pmatrix} 1 & \overline{\sigma} \\ \sigma & |\sigma|^2 \end{pmatrix} + \overline{\xi}_0 - u,$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad \sigma(\tau) = \frac{\beta(\tau)}{\alpha(\tau)},\quad \tau = \omega(\xi_0)$$

α, β are two holomorphic functions, ξ_0 is a complex constant, κ is a real number, u – scalar function. This solution is the localized at some values of the parameters of this model, and the length of the Higgs field is given by the following form

$$-tr\overline{\Phi}^2 = \tag{8}$$

$$= \frac{8r^4 + 32\kappa(x+t)(x^2-t^2)r^2 + 16\kappa r^4(x+t)}{((r^2+x^2-t)^2 + 2x^2 + 2t^2 + 1)^2} - 2\kappa^2.$$

With the help of the Darboux's transformation method we find the soliton solution for the simplest non-Abelian group $SU(2)$. Let

$$\widetilde{\psi} = (\xi - u)R\psi - T\psi, \tag{9}$$

where $R(u,v,r)$ and $T(u,v,r)$ are $2x2$ matrices and R is invertible, then the transformation $\psi \to \widetilde{\psi}$ is called a Darboux transformation of degree one if there are $(\widetilde{A}_\mu, \widetilde{\Phi})$ are satisfying

$$(r\widetilde{D}_r + \widetilde{\Phi} - 2(\xi - u)\widetilde{D}_u)\widetilde{\psi} = 0,$$

$$(r\widetilde{D}_v + \frac{\xi - u}{r}\widetilde{D}_r - \frac{\xi - u}{r^2}\widetilde{\Phi})\widetilde{\psi} = 0, \tag{10}$$

where $\widetilde{D}_\mu \psi = \partial_\mu \psi + \widetilde{A}_\mu \psi$. For given $(A, \Phi), (\widetilde{A}, \widetilde{\Phi})$ and arbitrary matrix function Q, let

$$\Delta_u R = 0,\quad r\Delta_r R + 2\Delta_u T + \delta R + 2R = 0,$$
$$r\Delta_r T + \delta T = 0.$$

Expressed in ψ, both equations of (10) are polynomials of ξ of degree two. The coefficients of the second, first and zero-th order of ξ in the two equations of (10) lead to four equations. After some transformations we give

$$\widetilde{A}_u = RA_u R^{-1} - (\partial_u R)R^{-1};$$
$$\widetilde{A}_v = TA_v T^{-1} - (\partial_v T)T^{-1}; \tag{11.1}$$

$$\widetilde{A}_r = \frac{1}{2}(TA_r - \partial T)T^{-1} + \frac{1}{2}(RA_r - \partial R)R^{-1} +$$
$$+ \frac{1}{2r}(T\Phi T^{-1} - R\Phi R^{-1}); \tag{11.2}$$

$$\widetilde{\Phi} = \frac{r}{2}(TA_r - \partial T)T^{-1} - \frac{r}{2}(RA_r - \partial R)R^{-1} +$$
$$+ \frac{1}{2}(T\Phi T^{-1} - R\Phi R^{-1}). \tag{11.3}$$

Take a trivial solution in the following form $A_\mu = 0 (\mu = u, v, r), \Phi = \kappa\sigma_3$. We can choose $R=I$, $T=S$. Then from (11.1)-(11.3), using the Theorem 1 [4] and after the some transformation we give a soliton solution in the form (7). Take ξ_0 to be a complex constant which is not real,

$$Z = \begin{pmatrix} \xi_0 & 0 \\ 0 & \overline{\xi}_0 \end{pmatrix}. \tag{12}$$

Let $\tau = \omega(\xi_0)$, then

$$H = \begin{pmatrix} \alpha(\tau) & -\overline{\beta}(\tau) \\ \beta(\tau) & \overline{\alpha}(\tau) \end{pmatrix}. \tag{13}$$

The matrix S has the following explicit form

$$S = \frac{\xi_0 - \overline{\xi}_0}{1+|\sigma|^2}\begin{pmatrix} 1 & \overline{\sigma} \\ \sigma & |\sigma|^2 \end{pmatrix} + (\overline{\xi}_0 - u)I, \tag{14}$$

$$\partial_u S = \frac{\xi_0 - \overline{\xi}_0}{1+|\sigma|^2}\begin{pmatrix} -(|\sigma|^2)_u & \overline{\sigma}_u - \overline{\sigma}^2\sigma_u \\ \sigma_u - \overline{\sigma}_u\sigma^2 & (|\sigma|^2)_u \end{pmatrix} - I. \tag{15}$$

We use (15) and obtain the following

$$\Phi^{(1)} = \frac{\xi_0 - \bar{\xi}_0}{(1+|\sigma|^2)^2} \times$$

$$\times \begin{pmatrix} (|\sigma|^2)_u & \bar{\sigma}^2\sigma_u - \bar{\sigma}_u \\ \bar{\sigma}_u\sigma^2 - \sigma_u & -(|\sigma|^2)_u \end{pmatrix} + \kappa \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (16)$$

We introduce

$$\xi_0 = i,\ \tau = \omega(\xi_0),\ \sigma(\tau) = \tau. \quad (17)$$

Then we have from (17)

$$\tau_u = (\nu - r^2(\xi_0 - u)^{-1})_u = -\frac{r^2}{(\xi_0 - u)^2};$$

$$\bar{\tau}_u = (\nu - r^2(\bar{\xi}_0 - u)^{-1})_u = -\frac{r^2}{(\bar{\xi}_0 - u)^2}. \quad (18)$$

We can calculate

$$tr(\Phi^{(1)})^2 = \frac{2(\xi_0 - \bar{\xi}_0)^2}{(1+|\sigma|^2)^4} \times$$

$$\times \{(|\sigma|^2)_u^2 + (\bar{\sigma}^2\sigma_u - \bar{\sigma}_u)(\sigma^2\bar{\sigma}_u - \sigma_u)\} +$$

$$+ \frac{4(\xi_0 - \bar{\xi}_0)\kappa}{(1+|\sigma|^2)^2}(|\sigma|^2)_u + 2\kappa^2. \quad (19)$$

We notice, that

$$\{(|\sigma|^2)_u^2 + (\bar{\sigma}^2\sigma_u - \bar{\sigma}_u)(\sigma^2\bar{\sigma}_u - \sigma_u)\} =$$

$$= \sigma_u\bar{\sigma}_u(1+|\sigma|^2)^2;$$

$$(|\sigma|^2)_u = (\sigma\bar{\sigma})_u = \sigma_u\bar{\sigma} + \sigma\bar{\sigma}_u,$$

so

$$tr(\Phi^{(1)})^2 = \frac{2(\xi_0 - \bar{\xi}_0)^2}{(1+|\sigma|^2)^4}\sigma_u\bar{\sigma}_u(1+|\sigma|^2)^2 +$$

$$+ \frac{4(\xi_0 - \bar{\xi}_0)\kappa}{(1+|\sigma|^2)^2}(\sigma_u\bar{\sigma} + \sigma\bar{\sigma}_u) + 2\kappa^2. \quad (20)$$

It is the localized solution and the length of the Higgs field is given by the form (8).

We conclude, if the Yang-Mills-Higgs equation has a trivial solution in the certain domain Ω, so the equation has onesoliton solution in the domain Ω with a constant spectral parameter. This solution is the localized solution at some values of the parameters of this model, and we can determine the length of the Higgs field.

The purpose of this paper is to deal with construction exact solutions in high dimension, especially for (2+1)-dimensional anti de Sitter space-time.

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Резюме

Классикалық және кванттық физикадағы әртүрлі өрістер теорияларының маңызды аспектісінің бірі – солитонды шешімдер. Интегралдану түсінігі арқылы солитондар жүйелеріне пайдалы классификация жүргізуге болады. Солитондық шешімдер бір-біріне тигенде қарапайым күйді және тұрақтылықты сақтайды. Есептің кері таралу әдісі сияқты әдістерді қолданып біз сызықты емес теңдеулердің көпсолитонды шешімдерін нақты түрде құра аламыз.

Резюме

Солитонные решения являются одним из важнейших аспектов многочисленных теории полей как в классической, так и в квантовой физике. Понятие интегрируемости позволяет сделать полезную классификацию солитонных систем. Солитонные решения этих систем имеют простое поведение при соударении и остаются устойчивыми. Используя такие методы как метод обратной задачи рассеяния, мы можем построить многосолитонные решения нелинейных уравнений в явном виде.

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