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ON SOME GENERALIZATIONS OF WEAK O-MINIMALITY ON PARTIAL ORDERS

Abstract. The article relates to o-minimality. A partially ordered structure is called weakly o-minimal if any definable subset is a finite union of convex sets. On how we can define a notion of a convex sets for a partial order the notion of weak o-minimality depends.

Keywords: o-minimal, partial order

Тірек сөздер: о-минималды, кейбір жинақтаулар.

Ключевые слова: о-минимальность, частичные порядки.

In [1] van den Dries considered o-minimal expansions of the ordered field of reals. Later in [2–4] Anand Pillay and Charles Steinhorn introduced a general notion of o-minimality. This notion became a very fruitful for investigating ordered structures. After that Max Dickmann consider an example of a weakly o-minimal structure. And then Dugald Macpherson, David Marker, and Charles Steinhorn developed a theory of weakly o-minimal structures, where they proved a lot of facts on weakly o-minimal structures, among them some properties of definable unary functions and definable subsets.

Definition. A set *A* is called a convex set if for all $a, b \in A$ it holds that

$$(a < b \to \forall x(a < x < b \to x \in A))$$

Definition (Anand Pillay, Charles Steinhorn)**.** A totally ordered structure ($M, <, \ldots$) is called *o-minimal* if any definable subset is a finite union of intervals and points

Definition (Macpherson, Marker, Steinhorn)**.** A totally ordered structure ($M, <, \ldots$) is called *weakly o-minimal* if any definable subset is a finite union of convex sets.

Definition. A set *A* is *strongly convex* if $A = \bigcup_{i \in I} B_i$ or $A = \bigcap_{j \in J} C_j$,where B_i, C_j are of the form (a,b), (a,b], [a,b), [a,b] and $B_i \subseteq B_j$ for $i < j$ and $C_i \supseteq C_j$ for $i < j$.

Example.

Let $\mathrm{M} = \mathrm{Q_a} \cup \mathrm{Q_b} \cup \mathrm{Q_c} \cup \mathrm{Q_d}$ where Q_* is a copy of the set of all rational number **Q** and for any $\mathrm{a_q} \in \mathrm{Q_a}, \mathrm{b_q} \in \mathrm{Q_b}, \mathrm{c_q} \in \mathrm{Q_c}, \mathrm{d_q} \in \mathrm{Q_d}$ we put that the following inequalities holds: $\mathrm{c_q} < \mathrm{a_q}, \mathrm{c_q} < \mathrm{b_q}, \mathrm{a_q} < \mathrm{d_q}, \mathrm{b_q} < \mathrm{d_q}$ and $\mathrm{a_q}, \mathrm{b_q}$ are incomparable.

Claim 1. There exist a convex set which we can define as the union of intervals, but we cannot define as the intersection of intervals and there exist a convex set which we can define as the intersection of intervals, but we cannot define as the union of intervals.

Proof is obvious.

Claim 2. Strongly convex set is convex.

Proof: Let A be strongly convex

Case1. $\mathrm{A} = \bigcup_{\mathrm{i} \in \mathrm{I}} \mathrm{B_i}$

Let $\mathrm{a}, \mathrm{b} \in \mathrm{A}$ be such that $\mathrm{a} < b$. Then $\mathrm{a} \in \mathrm{B_i}, \mathrm{b} \in \mathrm{B_j}$. Let $\mathrm{k} = \max(\mathrm{i}, \mathrm{j})$. Then $\mathrm{B_i} \subseteq \mathrm{B_k}, \mathrm{B_j} \subseteq \mathrm{B_k}$, so $\mathrm{a}, \mathrm{b} \in \mathrm{B_k}$, since $\mathrm{B_k}$ is an interval, then $(\mathrm{a}, \mathrm{b}) \subseteq \mathrm{B_k}$.

Case2. $A = \bigcap_{i\in I} B_i$

Let $a, b \in A$ be such that for each i $a < b$. Then $a \in B_i, b \in B_j$. Since B_i is an interval, then$(a, b) \subseteq B_k$ for each $i \Rightarrow (a, b) \subseteq \bigcap_{i\in I} B_i$

Definition. A partially ordered structure *(M,<,...)* is called *nearly po-minimal* if any definable subset is a finite union of strongly convex subsets.

Lemma 1. Any nearly po-minimal structure is *weakly po-minimal*.
Proof is obvious.

Lemma 2. Let *M* be nearly po-minimal. Assume that an interval (a, b) does contain a chain of length 2 and *M* is a lattice. Then (a, b) is finite.

Proof: Assume the contrary, that(a, b) is infinite then it contain an infinite antichain $\{c_1, c_2, \dots, c_n, \dots\}$. Note that $C_i \cup C_j = b$ and $C_i \cap C_j = a$ for any $i \neq j$, where b is exact upper bound and a is exact lower bound.

Let $\Psi(x, c_1, a, b) = a < x < b \cap x \neq c_1$. Then $\Psi(M, c_1, a, b)$ is a finite union of convex sets $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_n$. Let $\mathcal{V}_1 = \cup B_i$. Then $B_i \subseteq (a, b)$ so $B_i = (a, b)$ for $i > i_0$ for somei_0. But $c_1 \in (a, b)$ and $c_1 \notin \Psi(M, c_1, a, b)$ for a contradiction.

Let $\mathcal{V}_1 = \cap B_i$.

Since $\Psi(M, c_1, a, b)$ is infinite at least one of $\mathcal{V}_1$is infinite up to renumeration we may say that $\mathcal{V}_1$ is infinite. Then it contains c_i, c_j for some $i \neq j$. B_i is an interval, which contains c_i, c_j. Then it contains $a = c_i \cap c_j$ and $b = c_i \cup c_j$ Then $(a, b) \subseteq B_i \Rightarrow (a, b) = \cap B_i \Rightarrow c_1 \in c_1 \in \mathcal{V}_1 \Rightarrow c_1 \in \Psi(M, c_1, a, b)$ for a contradiction.

Lemma 3. There is a weakly o-minimal structure which is not nearly po-minimal.

Proof. Let $(N \cup \{\infty\}, \leq)$ be ordered as $0 <_1 n <_1 \infty$ for any positive n and n, k are incomparable for $0 < n \neq k < \infty$. Then this structure is a weakly po-minimal lattice but not nearly po-minimal.

REFERENCES

1 L. van den Dries. O-minimal structures, pp. 137-185 in Logic: from foundations to applications (Staffordshire, 1993 / Eedited by W. Hodges et al., Oxford Univ. Press, New York, 1996.

2 Pillay A., Steinhorn C. Definable sets in ordered structures, I. Trans. Amer. Math. Soc. 295:2 (1986), 565-592.

3 Knight J.F., Pillay A., Steinhorn C. Definable sets in ordered structures, II. Trans. Amer. Math. Soc. 295:2 (1986), 593-605.

4 Pillay A., Steinhorn C. Definable sets in ordered structures, III. Trans. Amer. Math. Soc. 309:2 (1988), 469-476.

5 Dickmann M.A. Elimination of quantifiers for ordered valuation rings, pp. 64-88 in Proceedings of the third Easter conference on model theory (Gross Koris, 1985),

Fachbereich Mathematik, Humboldt Univ. Berlin, 1985.
6 Macpherson D., Marker D., Steinhorn Ch. Weakly o-minimal structures and real closed fields. Transactions of the American Mathematical Society. 2000. Vol. 352. P. 5435–83.

Резюме

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БОСАҢ ОМИНИМАЛДЫ ЖАРТЫЛАЙ ТӘРТІБІНІҢ КЕЙБІР ЖИНАҚТАУЛАРЫ

Мақалада қатынас о-минималды жартылай тәртібінің кейбір жинақтаулары қарастырылған. Томпақ терімнің шектілік бірлестігі, ұғымын танысу білеміз.
Тірек сөздер: о-минималды, кейбір жинақтаулар.

Резюме

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НЕКОТОРЫЕ ОБОБЩЕНИЯ СЛАБОЙ О-МИНИМАЛЬНОСТИ НА ЧАСТИЧНЫЕ ПОРЯДКИ

В статье изучены некоторые обобщения слабой о-минимальности на частичные порядки. Частично заказанная структура называемая хилой о-минимальностью, если любое определимое подмножество – конечное объединение выпуклых наборов. Изучено понятие выпуклых наборов.
Ключевые слова: о-минимальность, частичные порядки.

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