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The Integral Error Functions Method for solving Heat equation and its application

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Abstract. Analytical solution of automodel heat transfer problem is represented by Integral Error Functions method. We observe that proposed method nicely fits the real life problem which is considered in the paper.

Introduction

It is Hartree 1935 who studied properties of Integral Error Function and reasonably sometimes these functions are called Hartree functions. We follow the method proposed by S.N. Kharin which is represented in [1], [2] and can be effectively used in diverse electric contact phenomena as it was shown in [3], [4].

Integral Error Functions and its properties

The integral error functions determined by recurrent formulas

$$i^n erfcx = \int_x^\infty i^{n-1} erfcv dv, \quad n=1,2,\ldots \qquad i^0 erfcx \equiv erfcx = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-v^2) dv \tag{1}$$

where

$$erfx = 1 - erfcx = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-v^2) dv \tag{2}$$

It is well known that the Integral Error Functions

$$u_n(\pm x, t) = t^{\frac{n}{2}} i^n erfc \frac{\pm x}{2a\sqrt{t}} \tag{3}$$

exactly satisfy the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{4}$$

and by superposition principle, linear combination of (3) or even series also satisfy (4)

$$u(x,t) = \sum_{n=0}^{\infty} [A_n u_n(x,t) + B_n u_n(-x,t)] \tag{5}$$

We consider (4) and solution (5) in degenerate domain where constants A_n, B_n have to be determined and can be derived by substituting (5) into boundary conditions if given boundary functions can be expanded into Taylor series with powers t or $\sqrt{t}$.

1. Using formula for Hermite polynomials one can derive

$$i^n erfc(-x) + (-1)^n i^n erfcx = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2m}}{2^{2m-1} m!(n-2m)!} \tag{6}$$

and represent (5) in the form of heat polynomials

$$u(x,t)=\sum_{n=0}^{\infty}A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^{2m}\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^{2m}\beta_{2n+1,m} \quad (7)$$

where

$$\beta_{n,m}=\frac{1}{2^{n+m-1}\cdot m!\cdot(n-2m)!} \quad (8)$$

2. Using L'Hopital rule it is not difficult to show that

$$\lim_{x\to\infty}\frac{i^n erfc(-x)}{x^n}=\frac{2}{n!} \quad (9)$$

Problem statement

The mathematical model of the temperature distribution in a copper semi-infinite bar with zero initial temperature and the entering heat flux density $P_0(t)=k+b\sqrt{t}$ where, $k=2\cdot10^{10}w\cdot m^{-2}\cdot k^{-1}$, $b=5\cdot10^{11}w\cdot m^{-2}\cdot k^{-1}sec^{-\frac{1}{2}}, a=9,4\cdot10^{-3}m\cdot s^{-0,5}$ and where also the time of melting point has to be found is represented as following automodel heat transfer problem.

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2} \qquad 0<x<\infty \quad (10)$$

$$t=0: \qquad u(\mathrm{x},0)=0 \quad (11)$$

$$x=0: \qquad -\lambda\frac{\partial u(0,t)}{\partial x}=P_0(t) \quad (12)$$

$$x=\infty: \qquad u(\infty,\mathrm{t})=0 \quad (13)$$

which can be solved by heat potential of single layer

$$u(x,t)=\int_0^t\frac{a\cdot e^{-\frac{x^2}{4a^2(t-\tau)}}}{\sqrt{\pi(t-\tau)}}\cdot\mu(\tau)d\tau$$

or by any classical method like Laplace transform etc.

Problem solution:

We represent solution in the following form:

$$u(x,t)=\sum_{n=0}^{\infty}A_n\left(2a\sqrt{t}\right)^n i^n erfc\left(\frac{x}{2a\sqrt{t}}\right) \quad (14)$$

where coefficients A_n have to be found.

$$u_x(0,t)=\lambda\sum_{n=0}^{2}A_n\left(2a\sqrt{t}\right)^{n-1}i^{n-1}erfc(0)=P_0(t) \quad (15)$$

$$u_x(0,t)=\frac{\lambda A_0 i^{-1}erfc(0)}{2a\sqrt{t}}+\lambda A_1 erfc(0)+\lambda 2a\sqrt{t}A_2 ierfc(0)=k+b\sqrt{t} \quad (16)$$

$$t^{-\frac{1}{2}}: \qquad \frac{\lambda A_0 i^{-1}erfc(0)}{2a\sqrt{t}}=0 \qquad A_0=0 \quad (17)$$

$$t^0: \qquad \lambda A_1 erfc(0)=k \qquad A_1=\frac{k}{\lambda erfc(0)} \quad (18)$$

$$t^{\frac{1}{2}}: \qquad \lambda 2a\sqrt{t}A_2 ierfc(0)=b\sqrt{t} \qquad A_2=\frac{b}{\lambda 2a ierfc(0)} \quad (19)$$

$$u(0,t)=\frac{k2a\sqrt{t}}{\lambda erfc(0)}ierfc(0)+\frac{b2at}{\lambda ierfc(0)}i^2erfc(0)=u_m \quad (20)$$