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ON SOLVABILITY OF LINEAR BOUNDARY VALUE PROBLEM FOR FREDHOLM

INTEGRO-DIFFERENTIAL EQUATION WITH PARAMETER

Abstract. By parameterization method a solvability criteria for the linear two-point boundary value problem for the Fredholm integro-differential equation containing parameter is established.

Ключевые слова: краевая задача, интегро-дифференциальное уравнение Фредгольма, параметр, разрешимость.

Тірек сөздер: шеттік есеп, Фредгольм интегралдық-дифференциалдық теңдеуі, параметр, шешілімділік.

Keywords: boundary value problem, Fredholm integro-differential equation, parameter, solvability.

Consider the linear two-point boundary value problem for integro-differential equation with parameter

$$\frac{dx}{dt} = A(t)x + \int_0^T K(t,s)x(s)ds + B(t)\lambda_0 + f(t), \quad t \in (0,T), \; x \in R^n, \; \lambda_0 \in R^m, \tag{1}$$

$$C_1 x(0) + C_2 x(T) = d, \quad d \in R^{n+m}, \tag{2}$$

where the $(n \times n)$ matrix $A(t)$, $(n \times m)$ matrix $B(t)$ and n vector $f(t)$ are continuous on $[0,T]$, the $(n \times n)$ matrix $K(t,s)$ is continuous on $[0,T]\times[0,T]$, $C_k : R^n \to R^{n+m}$, $k = 1,2$, $\|x\| = \max_{i=\overline{1,n}} |x_i|$.

Denote by $C([0,T], R^n)$ the space of continuous functions $x:[0,T] \to R^n$ with the norm $\|x\|_1 = \max_{t\in[0,T]} \|x(t)\|$.

Solution to problem (1), (2) is a pair $(\lambda_0^*, x^*(t))$, where $x^*(t) \in C([0,T], R^n)$ is a function, continuously differentiable on $(0,T)$, satisfying integro-differential equation (1) at $\lambda_0 = \lambda_0^*$ and boundary condition (2).

Works of many authors were devoted to the boundary value problems for differential and integro-differential equations, containing parameters (See [1-3] and references therein).

In [4] there proposed a method of investigation and solving the linear boundary value problem for Fredholm integro-differential equation. Necessary and sufficient conditions for the solvability of considered problem has been obtained, and the algorithm for finding its solution has been constructed.

The aim of present work is to establish the necessary and sufficient conditions for solvability of problem (1), (2).

For this purpose, we use the parametrization method [4]. Divide the interval $[0,T]$ into N parts with the step $h > 0 : Nh = T$. Denote the restriction of function $x(t)$ to the r-th interval $[(r-1)h, rh)$, $r = \overline{1,N}$, by $x_r(t)$. On introducing the additional parameters $\lambda_r \triangleq x_r((r-1)h)$ and making the substitute $u_r(t) = x_r(t) - \lambda_r$ on the r-th interval, we obtain the equivalent multi-point boundary value problem with parameters

$$\frac{du_r}{dt} = A(t)(u_r + \lambda_r) + \sum_{j=1}^{N} \int_{(j-1)h}^{jh} K(t,s)[u_j(s) + \lambda_j]ds + B(t)\lambda_0 + f(t), \tag{3}$$
$$t \in [(r-1)h, rh), \; r = \overline{1,N},$$

$$u_r[(r-1)h] = 0, \quad r = \overline{1,N}, \tag{4}$$

$$C_1\lambda_1 + C_2\lambda_N + C_2 \lim_{t\to T-0} u_N(t) = d, \quad d \in R^{n+m}, \tag{5}$$

$$\lambda_p + \lim_{t\to ph-0} u_p(t) - \lambda_{p+1} = 0, \quad p = \overline{1,N-1}, \tag{6}$$

where (6) are the conditions of continuity of solution at the interior partition points of interval $[0,T]$.

By $C([0,T],h,R^{nN})$ denote the space of function system $u[t]=(u_1(t),u_2(t),...,u_N(t))$, where $u_r:[(r-1)h,rh)\to R^n$ is continuous, and given all $r=\overline{1,N}$ it has the finite left-sided limit $\lim\limits_{t\to rh-0}u_r(t)$, with the norm $\|u[\cdot]\|_2=\max\limits_{r=\overline{1,N}}\sup\limits_{t\in[(r-1)h,rh)}\|u_r(t)\|$.

If the pair $(\lambda^*,u^*[t])$, where $\lambda^*=(\lambda_0^*,\lambda_1^*,\lambda_2^*,...,\lambda_N^*)\in R^{nN+m}$, $u^*[t]=(u_1^*(t),u_2^*(t),...,u_N^*(t))\in C([0,T],h,R^{nN})$ is a solution to problem (3) – (6), then the pair $(\lambda_0^*,x^*(t))$, where the function $x^*(t)$ is defined by the equalities: $x^*(t)=\lambda_r^*+u_r^*(t),\ \ t\in[(r-1)h,rh),\ \ r=\overline{1,N}$, $x^*(T)=\lambda_N^*+\lim\limits_{t\to T-0}u_N^*(t)$,

is a solution to initial problem (1), (2).

Suppose, that $X_r(t)$ is a fundamental matrix of differential equation $\dfrac{dx}{dt}=A(t)x$ on $[(r-1)h,rh]$, $r=\overline{1,N}$. Then the special Cauchy problem for system of integro-differential equations with parameters (3), (4) is equivalent to the system of integral equations

$$u_r(t)=X_r(t)\int\limits_{(r-1)h}^{t}X_r^{-1}(\tau_1)A(\tau_1)d\tau_1\lambda_r+X_r(t)\int\limits_{(r-1)h}^{t}X_r^{-1}(\tau_1)\times$$
$$\times\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh}K(\tau_1,s)[u_j(s)+\lambda_j]dsd\tau_1+X_r(t)\int\limits_{(r-1)h}^{t}X_r^{-1}(\tau_1)B(\tau_1)d\tau_1\lambda_0+$$
$$+X_r(t)\int\limits_{(r-1)h}^{t}X_r^{-1}(\tau_1)f(\tau_1)d\tau_1,\quad t\in[(r-1)h,rh),\ \ r=\overline{1,N}. \tag{7}$$

Assume in (7) that $t=\tau$. On multiplying both sides by $K(t,\tau)$, then integrating by τ on the interval $[(r-1)h,rh]$ and summing up over r, we have

$$\sum_{r=1}^{N}\int\limits_{(r-1)h}^{rh}K(t,\tau)u_r(\tau)d\tau=\sum_{r=1}^{N}\int\limits_{(r-1)h}^{rh}K(t,\tau)X_r(\tau)\int\limits_{(r-1)h}^{\tau}X_r^{-1}(\tau_1)A(\tau_1)d\tau_1d\tau\lambda_r+$$
$$+\sum_{r=1}^{N}\int\limits_{(r-1)h}^{rh}K(t,\tau)X_r(\tau)\int\limits_{(r-1)h}^{\tau}X_r^{-1}(\tau_1)\sum_{j=1}^{N}\int\limits_{(j-1)h}^{jh}K(\tau_1,s)[u_j(s)+\lambda_j]dsd\tau_1d\tau+$$
$$+\sum_{r=1}^{N}\int\limits_{(r-1)h}^{rh}K(t,\tau)X_r(\tau)\int\limits_{(r-1)h}^{\tau}X_r^{-1}(\tau_1)B(\tau_1)d\tau_1d\tau\lambda_0+$$
$$+\sum_{r=1}^{N}\int\limits_{(r-1)h}^{rh}K(t,\tau)X_r(\tau)\int\limits_{(r-1)h}^{\tau}X_r^{-1}(\tau_1)f(\tau_1)d\tau_1d\tau,\quad t\in[0,T]. \tag{8}$$

Introduce the following notations:

$\Phi_h(t)=\sum\limits_{j=1}^{N}\int\limits_{(j-1)h}^{jh}K(t,s)u_j(s)ds$, $M_0(h,t)=\sum\limits_{j=1}^{N}\int\limits_{(j-1)h}^{jh}K(t,\tau)X_j(\tau)\int\limits_{(j-1)h}^{\tau}X_j^{-1}(\tau_1)B(\tau_1)d\tau_1d\tau$,

$$M_r(h,t) = \int\limits_{(r-1)h}^{rh} K(t,\tau)X_r(\tau) \int\limits_{(r-1)h}^{\tau} X_r^{-1}(\tau_1)A(\tau_1)d\tau_1 d\tau +$$

$$+\sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K(t,\tau)X_j(\tau) \int\limits_{(j-1)h}^{\tau} X_j^{-1}(\tau_1) \int\limits_{(r-1)h}^{rh} K(\tau_1,s)dsd\tau_1 d\tau, \ r=\overline{1,N},$$

$$F(h,t) = \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K(t,\tau)X_j(\tau) \int\limits_{(j-1)h}^{\tau} X_j^{-1}(\tau_1)f(\tau_1)d\tau_1 d\tau.$$

Write down equation (8) in the form

$$\Phi_h(t) = \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K(t,\tau)X_j(\tau) \int\limits_{(j-1)h}^{\tau} X_j^{-1}(\tau_1)\Phi_h(\tau_1)d\tau_1 d\tau + \\ + \sum_{r=0}^{N} M_r(h,t)\lambda_r + F(h,t), \quad t\in[0,T]. \tag{9}$$

Choose the number $h_0 > 0$ satisfying the inequality

$$\beta T h_0 e^{\alpha h_0} < 1,$$

where $\beta = \max\limits_{(t,s)\in[0,T]\times[0,T]} \|K(t,s)\|$, $\alpha = \max\limits_{t\in[0,T]} \|A(t)\|$.

Using estimate (16) from [4, p. 1152] one easily may establish that for any $h \in (0,h_0]: Nh = T$ the integral equation (9) has a unique solution, and it can be found by the method of successive approximations.

By equalities

$$M_r^{(0)}(h,t) = M_r(h,t), \ M_r^{(k)}(h,t) = \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K(t,\tau)X_j(\tau) \int\limits_{(j-1)h}^{\tau} X_j^{-1}(\tau_1)M_r^{(k-1)}(h,\tau_1)d\tau_1 d\tau,$$

$$F^{(0)}(h,t) = F(h,t),$$

$$F^{(k)}(h,t) = \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K(t,\tau)X_j(\tau) \int\limits_{(j-1)h}^{\tau} X_j^{-1}(\tau_1)F^{(k-1)}(h,\tau_1)d\tau_1 d\tau, \quad k=1,2,...$$

we determine the sequences of matrices and vectors depending on $t\in[0,T]$. For $h \in (0,h_0]: Nh = T$ the unique solution to integral equation (9) can be represented in the form

$$\Phi_h(t) = \sum_{r=0}^{N} D_r(h,t)\lambda_r + F_h(t), \quad t\in[0,T], \tag{10}$$

where $D_r(h,t) = \sum\limits_{k=0}^{\infty} M_r^{(k)}(h,t)$ и $F_h(t) = \sum\limits_{k=0}^{\infty} F^{(k)}(h,t)$.

On substituting the right-hand side of (10) into (7) instead of $\sum\limits_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K(t,s)u_j(s)ds$, we obtain the representations of function $u_r(t)$ via λ_j and $f(t)$:

$$u_r(t) = X_r(t) \int\limits_{(r-1)h}^{t} X_r^{-1}(\tau)A(\tau)d\tau\lambda_r + \sum_{j=1}^{N} X_r(t) \int\limits_{(r-1)h}^{t} X_r^{-1}(\tau)\times$$

$$\times\left[D_j(h,\tau)+\int_{(j-1)h}^{jh}K(\tau,s)ds\right]d\tau\lambda_j+X_r(t)\int_{(r-1)h}^{t}X_r^{-1}(\tau)\big[D_0(h,\tau)+B(\tau)\big]d\tau\lambda_0+$$

$$+X_r(t)\int_{(r-1)h}^{t}X_r^{-1}(\tau)[F_h(\tau)+f(\tau)]d\tau,\quad t\in[(r-1)h,rh),\quad r=\overline{1,N}. \qquad (11)$$

Find $\lim_{t\to T-0}u_N(t)$ and $\lim_{t\to ph-0}u_p(t),\ \ p=\overline{1,N-1}$ from (11). Substituting their corresponding expressions into boundary condition (5) and bonding condition (6), we get the system of linear algebraic equations

$$C_2X_N(T)\int_{T-h}^{T}X_N^{-1}(\tau)\big[D_0(h,\tau)+B(\tau)\big]d\tau\lambda_0+$$

$$+\left\{C_1+C_2X_N(T)\int_{T-h}^{T}X_N^{-1}(\tau)\left[D_1(h,\tau)+\int_0^hK(\tau,s)ds\right]d\tau\right\}\lambda_1+$$

$$+C_2\sum_{i=1}^{N-1}X_N(T)\int_{T-h}^{T}X_N^{-1}(\tau)\left[D_i(h,\tau)+\int_{(i-1)h}^{ih}K(\tau,s)ds\right]d\tau\lambda_i+$$

$$+C_2\left\{I+X_N(T)\int_{T-h}^{T}X_N^{-1}(\tau)\left[A(\tau)+D_N(h,\tau)+\int_{T-h}^{T}K(\tau,s)ds\right]d\tau\right\}\lambda_N=$$

$$=d-C_2X_N(T)\int_{T-h}^{T}X_N^{-1}(\tau)[F_h(\tau)+f(\tau)]d\tau, \qquad (12)$$

$$X_p(ph)\int_{(p-1)h}^{ph}X_p^{-1}(\tau)\big[D_0(h,\tau)+B(\tau)\big]d\tau\lambda_0+$$

$$+\left\{I+X_p(ph)\int_{(p-1)h}^{ph}X_p^{-1}(\tau)\left[A(\tau)+D_p(h,\tau)+\int_{(p-1)h}^{ph}K(\tau,s)ds\right]d\tau\right\}\lambda_p-$$

$$-\left\{I-X_p(ph)\int_{(p-1)h}^{ph}X_p^{-1}(\tau)\left[D_{p+1}(h,\tau)+\int_{ph}^{(p+1)h}K(\tau,s)ds\right]d\tau\right\}\lambda_{p+1}+$$

$$+\sum_{\substack{j=1\\ j\neq p,\,j=p+1}}^{N}X_p(ph)\int_{(p-1)h}^{ph}X_p^{-1}(\tau)\left[D_j(h,\tau)+\int_{(j-1)h}^{jh}K(\tau,s)ds\right]d\tau\lambda_j=$$

$$=-X_p(ph)\int_{(p-1)h}^{ph}X_p^{-1}(\tau)\big[F_h(\tau)+f(\tau)\big]d\tau,\quad p=\overline{1,N-1}. \qquad (13)$$

Denote the matrix, corresponding to the left-hand side of the system of equations (12), (13) by $Q^*(h)$, and vector, corresponding to the right-hand side of the system of equations by $F^*(h)$. Then this system can be written as follows

$$Q^*(h)\lambda=F^*(h),\qquad \lambda\in R^{nN+m}. \qquad (14)$$

For any $h\in(0,h_0]:Nh=T$ the following assertion is true

Theorem. *Problem (1), (2) is solvable if and only if for any* $\eta \in Ker(Q^*(h))'$ *the equality* $(\eta, F^*(h)) = 0$ *holds, where* $(\cdot,\cdot)$ *is a scalar product in* R^{nN+m}*, i.e. when the right-hand side of equation (14) is orthogonal to the kernel of transposed matrix* $\left(Q^*(h)\right)'$.

Definition. Problem *(1), (2) is called uniquely solvable if for any pair* $\left(f(t),d\right)$*, where* $f(t) \in C\left(\left[0,T\right], R^n\right)$, $d \in R^{n+m}$*, it has a unique solution.*

Corollary. Problem *(1), (2) is uniquely solvable if and only if the matrix* $Q^*(h): R^{nN} \to R^{nN}$ *is invertible for* $\forall\, h \in \left(0, h_0\right]: Nh = T$.

REFERENCES

1 Luchka A.Yu., Nesterenko O.B. Construction of solutions of integro-differential equations with restrictions and control by projection-iterative method // Nonlinear Oscillations. – 2009. V. 12. № 1. P. 85-93.

2 Samoilenko A.M., Ronto N.I., Ronto B.A. Two-point boundary value problem with parameter in boundary conditions // Doklady AS Ukrainian SSR. Ser. Phys.-math. Techn. sciences. – 1985. – T. 7. – P. 22-26. (in russian).

3 Dzhumabaev D.S., Minglibayeva B.B. Well-posedness of linear two-point boundary value problem with parameter // Mathematical journal. – 2004. – T. 4. № 1. – P. 41-51. (in russian).

4 Dzhumabaev D.S. A method for solving the linear boundary value problem for an integro-differential equation // Computational mathematics and mathematical physics. – 2010. V. 50. № 7. P. 1150-1161.

Резюме

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ПАРАМЕТРІ БАР ФРЕДГОЛЬМ ИНТЕГРАЛДЫҚ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУІ ҮШІН СЫЗЫҚТЫ ШЕТТІК ЕСЕБІНІҢ ШЕШІЛІМДІЛІГІ ТУРАЛЫ

Параметрі бар Фредгольм интегралдық-дифференциалдық теңдеуі үшін сызықты екі нүктелі шеттік есептің шешілімділігінің критерий параметрлеу әдісі негізінде алынған.

Тірек сөздер: шеттік есеп, Фредгольм интегралдық-дифференциалдық теңдеуі, параметр, шешінділік.

Резюме

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О РАЗРЕШИМОСТИ ЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ФРЕДГОЛЬМА С ПАРАМЕТРОМ

Аннотация. Методом параметризации установлен критерий разрешимости линейной двухточечной краевой задачи для интегро-дифференциального уравнения Фредгольма, содержащего параметр.

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