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PERMUTATION S_n – MODULES IN FREE NOVIKOV ALGEBRAS

Abstract. We study permutation modules that are isomorphic to S_n-submodules of free Novikov algebras and give necessary conditions on permutation modules to be Novikov admissible.

Keywords: partition, S_n -module, Novikov algebra.
Ключевые слова: разбиение, Sn-модуль, алгебра Новикова.
Тірек сөздер: жіктелім, Sn-модуль, Новиков алгебрасы.

1. Introduction. Let n be a positive integer number. A partition of n is a sequence $\propto = (\propto_1, \propto_2, \dots, \propto_k)$ such that $n = \propto_1 + \dots + \propto_k$ and $\propto \geq \dots \geq \propto_k \geq 1$, $\propto_i$ is called a part of $\propto$ and k is the length of $\propto$. We write $\propto \vdash n$ if $\propto$ is a partition of n. Also, it is common in literatures to write a partition of n in the following form:

$$\propto = (n^{i_n}, \dots, 2^{i_2}, 1^{i_1})$$

where i_j is the number of occurrence of the integer j in the partition $\propto$. Let us denote by $P(n)$ the set of partitions of n.

Definition. For $\propto = (n^{i_n}, \dots, 2^{i_2}, 1^{i_1}) \in P(n)$ the partition $w(\propto) \in P(n+1)$ defined by

$$w(\propto) = sort(n + 1 - \sum_{j=1}^{n} i_j, i_1, i_2, \dots, i_n)$$

Is called the *weight* of $\propto$.

A weight of a partition is studied in [2]. Weights are used to describe irreducible S_n-module components of multilinear parts of free Novikov algebras and also showed that there are modules in free Novikov algebras which are isomorphic to permutation modules $M^{w(\propto)}$ corresponding to weights $w(\propto)$ for any $\propto \vdash n$. For more details about structures of S_n-modules and permutations modules see [4], [5] or [6]. In [3] considered two types of permutation modules in free Novikov algebras and given their decomposition into Specht modules.

But we do not know what a permutation module appears and what its multiplicity is in free Novikov algebra. These kind of questions motivated us to study combinatorial properties of weights and in our paper we find necessary conditions on partitions of $n+1$ to be a weight of partitions of n. More precisely, we calculate maximal length of weights and find minimal weight with respect to lexicographic order in the set of weights of partitions of n.

MAIN RESULTS

2. Main results. Theorem 2.1. *Maximal length of weights of partitions of n is equal to the number*

$$1 + \left\lfloor \frac{-1 + \sqrt{8n+1}}{2} \right\rfloor$$

Where $\lfloor a \rfloor$ is a integer part of a.

Proof. By definition of weight, we can say that length of weight of a partition equals $k+1$ if and only if the partition has k different parts. To obtain maximal length of weight, it is enough to consider a partition of n with k different parts in the following form:

$$\propto = (n - \frac{k(k-1)}{2}, k-1, \dots, 1)$$

so that

$$n - \frac{k(k-1)}{2} \geq k$$

By solving last inequality, we get the proof of our theorem.

Let $\propto = (\propto_1, \propto_2, \dots, \propto_k)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_k)$ are partitions of n. Recall that $\propto > \beta$, in lexicographic order if, for some index i,

$$\propto_j = \beta_j, \quad for\ j < i\ and\ \propto_i = \beta_i$$

One can check that lexicographic order is a total order in a set of partitions.

Theorem 2.2. *Let β be the minimal partition in the set of weights of partitions of n.*

If $n \equiv 0 (mod\ 3)$, then $\beta = (\frac{n+3}{3}, \frac{n}{3}, \frac{n}{3})$.

If $n \equiv 1 (mod\ 3)$, then $\beta = (\frac{n+2}{3}, \frac{n+2}{3}, \frac{n-1}{3})$.

If $n \equiv 2(mod\ 3)$, *then* $\beta = (\frac{n+4}{3}, \frac{n+1}{3}, \frac{n-5}{3}, 1)$.

Remark 2.3. *If we get a negative number as a part of a partition in Theorem 2.2, we remove this part from the given partition.*

Proof. In our proof we apply the following proposition:

Proposition 2.4. (Lemma 8.2. in [2]) *Let* $(\gamma_1, \gamma_2, \dots, \gamma_k) \vdash n+1$. *If* $\gamma_3 + 2\gamma_4 + \dots + (k-2)\gamma_k > \gamma_1 - 1$ *then* $\gamma \notin Im\, w$.

In order to prove the Theorem 2.2, we only consider the case when $n \equiv 0(mod\ 3)$, because other cases are proved similarly. Suppose that $\beta = (\frac{n+3}{3}, \frac{n}{3}, \frac{n}{3})$ is not a minimal in weight sets, so there is a partition $\propto$ in $P(n)$ so that $\beta > \gamma = (\gamma_1, \gamma_2, \dots, \gamma_k) = w(\propto)$.

Now we consider three possible cases for γ such that $\beta > \gamma$. They are following:

1) $\gamma_1 = \frac{n+3}{3}, \gamma_2 = \frac{n}{3},\ and\ \gamma_3 + \dots + \gamma_k = \frac{n}{3}$.

2) $\gamma_1 = \frac{n+3}{3},\ \frac{n}{3} > \gamma_2 \geq \dots \geq \gamma_k \geq 1$.

3) $\frac{n+3}{3} > \gamma_1$.

In all cases γ satisfies the inequality in proposition 2.4, so we get contradictions to our assumption of our lemma.

Let us endow partitions set $P(n)$ by equivalence relation. For $\propto, \beta \in P(n)$ say that $\propto \sim \beta$ if $w(\propto) = w(\beta)$. It is easy to see that this relation is reflexive, symmetric and transitive. Let $\tilde{P}(n)$ be the set of equivalence classes of $P(n)$ under this relation. Generating function for partitions $G_p(x) = \sum_i p_i x^i$, where $p_i = |P(i)|$, satisfies the following relation

$$G_p(x) = \prod_i (1 - x^i)^{-1}$$

It will be interesting to find similar relation for a generating function of factor-set $\tilde{P}(n)$. Beginning part of $G_{\tilde{p}}(x) = \sum_i \tilde{p}_i x^i$, where $\tilde{p}_i = |\tilde{P}(i)|$ look like

$$G_{\tilde{p}}(x) = x + x^2 + 2x^3 + 4x^4 + 3x^5 + 8x^6 + 7x^7 + 12x^8 + 13x^9 + 19x^{10} + \cdots$$

We see that coefficients are not monotone. But if we consider separately even degree parts and odd degree parts, coefficients are increasing

$$G_{\tilde{p}}^{even}(x) = x^2 + 4x^4 + 8x^6 + 12x^8 + 19x^{10} + 32x^{12} + 42x^{14} + 64x^{16}$$

$$G_{\tilde{p}}^{odd}(x) = x + 2x^3 + 3x^5 + 7x^7 + 13x^9 + 19x^{11} + 30x^{13} + 48x^{15} + 64x^{17} + 93x^{19} + 131x^{21} + \cdots$$

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Резюме

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ЕРКІН НОВИКОВ АЛГЕБРАЛАРЫНДАҒЫ АЛМАСТЫРМАЛЫ S_n – МОДУЛЬДЕРІ

Еркін Новиков алгебрасындағы алмастырмалы S_n – модульдері қарастырылған және еркін Новиков алгебрасында кездесетін алмастырмалы S_n – модульдері үшін қажетті шарттары анықталған.

Тірек сөздер: жіктелім, S_n – модуль, Новиков алгебрасы.

Резюме

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ПЕРЕСТАНОВОЧНЫЕ S_n – МОДУЛИ В СВОБОДНЫХ АЛГЕБРАХ НОВИКОВА

Изучаются перестановочные S_n – модули которые встречаются в свободных алгебр Новикова и определяются необходимые условия для таких модулей.

Ключевые слова: Разбиение, S_n – модуль, алгебра Новикова.

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