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CLARKSON WEAK MAJORIZATION INEQUALITIES FOR τ-MEASURABLE OPERATORS

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Abstract. Let (M,τ) be a semi-finite von Neumann algebra and f be a nonnegative function on $[0,\infty)$ with $f(0)=0$. Let $x_1, x_2, \cdots, x_n$ be τ-measurable operators and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be positive real numbers such that $\sum_{j=1}^{n} \alpha_j = 1$. We have the following results.

(1) If $g(t) = f(\sqrt{t})$ is convex, then

$$f\left(\left|\sum_{j=1}^{n}\alpha_j x_j\right|\right) + \sum_{1\le j<k\le n} f\left(\sqrt{\alpha_j\alpha_k}\left|x_j - x_k\right|\right) \preceq \sum_{j=1}^{n}\alpha_j f\left(\left|x_j\right|\right).$$

(2) If $h(t) = f(\sqrt{t})$ is concave, then

$$\sum_{j=1}^{n}\alpha_j f\left(\left|x_j\right|\right) \preceq f\left(\left|\sum_{j=1}^{n}\alpha_j x_j\right|\right) + \sum_{1\le j<k\le n} f\left(\sqrt{\alpha_j\alpha_k}\left|x_j - x_k\right|\right).$$

1. Introduction. Let M_n be von Neumann algebra of $n\times n$ complex matrices, and let M_n^+ be positive part of M_n. Hirzallah and Kittaneh in [5] proved the following noncommutative Clarkson inequalities for $n-$tuples of operators: Let $|||\cdot|||$ be a unitarily norm, $A_0,\cdots,A_{n-1}\in \mathrm{M}_n^+$ and $\alpha_0,\cdots,\alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1}\alpha_j=1$.

(1) If f be a nonnegative function on $[0,\infty)$ such that $f(0)=0$ and $g(t)=f(\sqrt{t})$ is convex on $[0,\infty)$, then

$$\left|\left|\left|f\left(\left|\sum_{j=0}^{n-1}\alpha_j A_j\right|\right)+\sum_{0\le j<k\le n-1} f\left(\sqrt{\alpha_j\alpha_k}\left|A_j-A_k\right|\right)\right|\right|\right|\le\left|\left|\left|\sum_{j=0}^{n-1}\alpha_j f\left(\left|A_j\right|\right)\right|\right|\right|.$$

(2) If f be a nonnegative function on $[0,\infty)$ such that $g(t)=f(\sqrt{t})$ is concave on $[0,\infty)$, then

$$\left|\left|\left|\sum_{j=0}^{n-1}\alpha_j f\left(\left|A_j\right|\right)\right|\right|\right|\le\left|\left|\left|f\left(\left|\sum_{j=0}^{n-1}\alpha_j A_j\right|\right)+\sum_{0\le j<k\le n-1} f\left(\sqrt{\alpha_j\alpha_k}\left|A_j-A_k\right|\right)\right|\right|\right|.$$

The purpose of this paper is to extend the above results to $n-$tuples of $\tau-$measurable operators.

This paper is organized as follows. Section 2 contains some preliminary definitions.

In section 3, we proved the weak majorization type of Clarcson inequalities for n tuples of $\tau-$measurable operators.

2. Preliminaries. Throughout this paper, we denote by M a semi-finite von Neumann algebra on the Hilbert space H with a normalized normal faithful finite trace τ. The closed densely defined linear operator x in H with domain $D(x)$ is said to be affiliated with M if and only if $u^*xu=x$ for all unitary u which belong to the commutant M' of M. If x is affiliated with M, the x said to be $\tau-$measurable if for every $\varepsilon>0$ there exists a projection $e\in M$ such that $e(H)\subseteq D(x)$ and $\tau(e^\perp)<\varepsilon$ (where for any projection e we let $e^\perp=1-e$). The set of all $\tau-$measure operators will be denoted by $L_0(M)$. The set $L_0(M)$ is a $*-$algebra with sum and product being the respective closure of the algebraic sum and product. Let $P(M)$ be the lattice of projections of M. The sets

$$\aleph(\varepsilon,\delta)=\left\{\ x\in L_0(M):\ \exists e\in P(M)\ \text{such that}\ \|xe\|<\varepsilon,\ \tau(e^\perp)<\delta\right\}$$

$(\varepsilon,\delta>0)$ from a base at 0 for an metrizable Hausdorff topology in $L_0(M)$ called the measure topology. Equipped with the measure topology, $L_0(M)$ is a complete topological $*-$algebra (see [6]). For $x\in L_0(M)$, the generalized singular value function $\mu(x)$ of x is defined by

$$\mu_s(x)=\inf\left\{\ \|xe\|:\ e\in P(M),\ \tau(e^\perp)\le s\right\}\quad (s\ge 0).$$

If $x,y\in L_0(M)$, then we say that x is submajorized by y and write $x\preceq y$ if and only if

$$\int_0^t\mu_s(x)ds\le\int_0^t\mu_s(y)ds,\quad \forall t>0.$$

As Proposition 4.6 in [3], we obtain the following result.

Lemma 2.1. Let f be a continuous increasing function on $\mathbf{R}_+$ with $f(0)=0$. Let $x_1,x_2,\cdots,x_n$ be positive $\tau-$measurable operators and let $a_1,a_2,\cdots,a_n$ be positive elements in M with $\sum_{j=1}^n a_j^*a_j\le 1$.