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MINIMAL ELEMENTS AND MINIMAL COVERS IN ROGERS SEMILATTICES

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Abstract. This article is devoted to studying algebraic properties of Rogers semilattices of generalized computable families of total functions. Indeed, it is a continuation of attempts to find the elementary properties of the corresponding Rogers semilattices, which differ from properties of the classical Rogers semilattices for families of computable functions. Families of total A-computable functions, where $\emptyset' \leq_T A$, will be considered. The basic idea was taken from [1].

Some definitions and notations. For a family F of total functions a numbering $\alpha : \omega \to F$ is called computable if the binary function $\lambda x \lambda y \alpha(x)(y)$ is total recursive. Let $\alpha : \omega \to F$ and $\beta : \omega \to F$ be two numberings of the same set F. We say that the numbering α is reducible to the numbering β, if there is a computable function f such that $\alpha = \beta \circ f$, and we write this symbolically as: $\alpha \leq \beta$. If $\alpha \leq \beta$ and $\beta \leq \alpha$ then the numberings α and β are said to be equivalent, written as $\alpha \equiv \beta$. Denote by $\deg(\alpha)$ the degree of α, that is, the set $\{\beta \mid \beta \equiv \alpha\}$ of numberings. The reducibility of numberings is a pre-order relation on the set of all computable numberings of a family F, which we denote by $Com(F)$, and it induces a partial order relation on a set of degrees of the numberings in $Com(F)$, which we also denote by $\leq$. The partially ordered set $\Re(F) = \langle \{\deg(\alpha) \mid \alpha \in Com(F)\}, \leq \rangle$ is an upper semilattice, which we call the Rogers semilattice of the family F.

A numbering α of a set F is called minimal if, for any numbering β of F, $\beta \leq \alpha$ implies that $\alpha \leq \beta$. A computable one-to-one numbering is called a Friedberg numbering. The numerical equivalence θ_α of a numbering α is defined as follows: $\theta_\alpha = \{(x, y) \mid \alpha(x) = \alpha(y)\}$. An equivalence relation η is said to be decidable (positive) if η is computable (computably enumerable). For the further undefined notions we refer to [2, 3].

Let F be a family of total functions which are computable by an oracle A. A numbering $\alpha : \omega \to F$ is called A-computable if the binary function $\alpha(n)(x)$ is A-computable, [4]. A family F is called A-computable if it has an A-computable numbering. If A is a recursive set, then we are dealing with a family of computable functions and its classical computable numberings. The partially ordered set $\Re_A(F) = \langle \{\deg(\alpha) \mid \alpha \in C_A(F)\}, \leq \rangle$, where $C_A(F)$ denotes the set of all A-computable numberings of the family F, is called the Rogers semilattice of the family F, [4].

It is known that the Rogers semilattice of every infinite Σ^0_{n+2}-computable family contains an infinite number of minimal elements, and if a family $S \subseteq \Sigma^0_{n+2}$ has a Σ^0_{n+2}-computable Friedberg numbering then S has an infinite number of non-equivalent positive undecidable Σ^0_{n+2}-computable numberings and an infinite number of non-equivalent Σ^0_{n+2}-computable minimal non-positive numberings, [1]. Generalizing these results according to [4] and doing some simplifying in their proofs we have the next statements.

Theorem 1 [5, 6]. Let F be an infinite A-computable family of total functions, where $\emptyset' \leq_T A$. Then F has infinitely many pairwise nonequivalent A-computable Friedberg numberings.

This theorem is obtained after generalization theorem 1 from [5].

Theorem 2. Let F be an infinite A-computable family of total functions, where $\emptyset' \leq_T A$. Then F has infinitely many pairwise nonequivalent positive non-decidable A-computable numberings.

Proof. By theorem 1, suppose that α is an A-computable Friedberg numbering of the family F. Let M be a maximal set and $\overline{M} = \{\overline{m}_0 < \overline{m}_1 < \overline{m}_2 < \ldots\}$. For all $k \in \omega$ we construct numberings β_k of the family F by the following way:

$$\beta_k(\overline{m}_i) = \alpha(i) \text{ for any } i \in \omega \text{, and } \beta_k(x) = \alpha(k) \text{ for any } x \in M.$$

Then

$$y = \beta_k(n)(x) \Leftrightarrow \big(n \in M \ \&\ y = \alpha(k)(x)\big) \ or \ \big(n \in \overline{M} \ \&\ \exists i\,(\overline{m}_i = n \ \&\ y = \alpha(i)(x))\big),$$

where the relations $n \in M$, $n \in \overline{M}$ and $\overline{m}_i = n$ are $\emptyset'$-computable. Therefore β_k is an A-computable numbering of the family F. Also

$$\beta_k(x) = \beta_k(y) \Leftrightarrow x = y \ or \ \big(x \in M \cup \{p\} \ \&\ y \in M \cup \{p\}\big),$$

where $p \in \overline{M}$ is a β_k-index of the function $\alpha(k)$. So, obviously, θ_{β_k} is a c.e. equivalence. Since the set $\beta_k^{-1}(\alpha(k)) = M \cup \{p\}$ is not computable the numbering β_k is not decidable.

Now we will show that if $k \neq l$ then $\beta_k \not\leq \beta_l$. Assume $\beta_k(x) = \beta_l(f(x))$ for some computable function f. Since $\beta_l(x) \neq \alpha(k)$ for any $x \in M$, it follows that $f(M) \subseteq f(\beta_k^{-1}(\alpha(k))) \subseteq \overline{M}$. And note that the set $\overline{M} \setminus f(\beta_k^{-1}(\alpha(k)))$ contains β_l-indices of all functions in F distinct from $\alpha(k)$, therefore this set is infinite. So, since M is maximal set, the c.e. set $f(M)$ is finite. Moreover, by computability of the set $f^{-1}(f(M)) = \{y \mid f(y) \in f(M)\}$, the set M is computable, which contradicts with maximality of the set M.

Theorem 3. Let F be an infinite A-computable family of total functions, where $\emptyset' \leq_T A$. Then F has infinitely many pairwise nonequivalent minimal non-positive A-computable numberings.

Proof. By theorem 1, suppose that α is an A-computable Friedberg numbering of the family F. Let M be a maximal set and $\overline{M} = \{\overline{m}_0 < \overline{m}_1 < \overline{m}_2 < \ldots\}$. By theorem 2, we know that if we define numbering β of the family F by the following way:

$$\beta(\overline{m}_i) = \alpha(i) \text{ for any } i \in \omega \text{, and } \beta(x) = \alpha(p) \text{ for any } x \in M \text{ with fixed } p \in \omega,$$

then β is positive non-decidable A-computable numbering of F such that $\beta^{-1}(\alpha(p))$ is a non-computable c.e. set. For all $k \in \omega$ we construct numberings γ_k of the family F by the next way:

$$\gamma_k(\overline{m}_i) = \beta(i) \text{ for any } i \in \omega \text{, and } \gamma_k(x) = \beta(k) \text{ for any } x \in M.$$

We will show that if $\beta(k) \neq \alpha(p)$ then γ_k is a minimal non-positive A-computable numbering of F. First, we prove that γ_k is a minimal numbering. Let γ be a numbering of F which is reducible to γ_k via some computable function f, where $rng(f) = W$. Since F is an infinite family and $\gamma_k(x) = \beta(k)$ for any $x \in M$, it follows that the set $W \cap \overline{M}$ is infinite. Then, by M is a maximal set,

$\overline{M} \setminus W$ is a finite set. Now we construct a computable function g reducing γ_k to γ. We enumerate the sets M and W simultaneously, and for every $x \in M \cup W$ if x appears in M first then $g(x)$ is equal to some fixed γ-index of the function $\beta(k)$ (for example, $g(x) = c = \mu_m(\mu_z(\hat{f}(z) = f(m)))$, where $\hat{f}$ is computable function such that $rng(\hat{f}) = M$), otherwise $g(x) = \mu_y(f(y) = x)$. Finally if x belongs to the finite set $\overline{M} \setminus W$ then again $g(x)$ is equal to some fixed γ-index of the function $\beta(k)$, which completes the construction.

Now it remains to show that if $\beta(k) \neq \alpha(p)$ then γ_k is a non-positive A-computable numbering. $\gamma_k(x) = \alpha(p)$ for all $x \in \hat{W} \subseteq \overline{M}$, since $\gamma_k(x) = \beta(k) \neq \alpha(p)$ for all $x \in M$, where $\hat{W} = \gamma_k^{-1}(\alpha(p))$ is infinite set. If γ_k were positive numbering then the set $\hat{W}$ would be c.e. set, it follows that $\omega \setminus \hat{W}$ is an infinite set since $M \subseteq \omega \setminus \hat{W}$, and $\hat{W} \setminus M$ is an infinite set since $\hat{W} \subseteq \overline{M}$, which contradicts with the affirmation that M is maximal set.

Theorem 4 [7]. Let F be an A-computable family of total functions, where $\emptyset' \leq_T A$. If F contains at least two functions then F has no A-computable principal numbering.

This theorem is obtained after generalization theorem 3 from [5].

If for any non-principal A-computable numbering α of a family S we can find an A-computable numbering β of S such that $\deg(\beta) > \deg(\alpha)$, and for every degree $\deg(\gamma)$, if $\deg(\alpha) \leq \deg(\gamma) \leq \deg(\beta)$ then either $\deg(\gamma) = \deg(\alpha)$ or $\deg(\gamma) = \deg(\beta)$, then $\deg(\beta)$ is called a minimal cover of $\deg(\alpha)$.

It is known that if a family $S \subseteq \Sigma^0_{n+2}$ contains at least two sets C and D, and α be a Σ^0_{n+2}-computable numbering of S, and if the set $\alpha^{-1}(C)$ is 0'-computable, then $\deg(\alpha)$ has a minimal cover, [1]. But we have the next statement for an A-computable family of total functions.

Theorem 5. Let F be an A-computable family of total functions and let α be an A-computable numbering of F, where $\emptyset' \leq_T A$. If F contains at least two functions then $\deg(\alpha)$ has a minimal cover.

Proof. By theorem 4, we know that any numbering of F is not principal. Let M be a maximal set and $\overline{M} = \{\overline{m}_0 < \overline{m}_1 < \overline{m}_2 < \ldots\}$. Then there exists an one-to-one computable function f such that $f(\omega) = M$. If we define $\beta(x) = \alpha(f^{-1}(x))$ for all $x \in M$ then $\alpha \leq \beta$ via f and β is an A-computable since $f^{-1}(x)$ is $\emptyset'$-computable set. Now we take $f_1, f_2 \in F$ and find a number $a \in \omega$ such that $f_1(a) \neq f_2(a)$. We complete construction β by defining values $\beta(x)$ for $x \in \overline{M}$. If $\varphi_e(\overline{m}_e)\downarrow$ and $\alpha(\varphi_e(\overline{m}_e))(a) = f_1(a)$ then we define $\beta(\overline{m}_e) = f_2$, otherwise $\beta(\overline{m}_e) = f_1$. It is clear that β is still an A-computable. We will show that $\beta \not\leq \alpha$. Suppose $\beta(x) = \alpha(\varphi_e(x))$, $x \in \omega$, for some computable function φ_e. Put $x = \overline{m}_e$. If $\alpha(\varphi_e(\overline{m}_e))(a) = f_1(a)$ then by construction $\beta(\overline{m}_e)(a) = f_2(a) \neq f_1(a) = \alpha(\varphi_e(\overline{m}_e))(a)$, contradiction. If $\alpha(\varphi_e(\overline{m}_e))(a) \neq f_1(a)$ then by construction $\beta(\overline{m}_e)(a) = f_1(a) \neq \alpha(\varphi_e(\overline{m}_e))(a)$, contradiction again. Therefore $\deg(\alpha) < \deg(\beta)$.

Now let γ be an A-computable numbering of F such that $\alpha \leq \gamma$ and $\gamma \leq \beta$. Suppose $\gamma \leq \beta$ via computable function h where $rng(h) = V$ then $\gamma = \beta h$, also we know that $\alpha = \beta f$ where $rng(f) = M$. Since $\beta f \leq \beta h \leq \beta$, it follows that we can assume $M \subseteq V \subseteq \omega$ (otherwise we can redefine the function h such that the condition is satisfied). Since the set M is maximal, either $\omega \setminus V$ or $V \setminus M$ is finite. In the former case (for obvious reasons) $\gamma \equiv \beta$, in the latter case $\gamma \equiv \alpha$.

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РОДЖЕРС ЖАРТЫТОРЛАРЫНДАҒЫ МИНИМАЛ ЭЛЕМЕНТТЕР ЖӘНЕ МИНИМАЛ ЖАБУЛАР

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Тірек сөздер: минимал нөмірлеу, позитив нөмірлеу, шешілімді нөмірлеу, минимал жабу.

Аннотация. Мақала барлық жерде анықталған функциялардың жалпыланған есептелімді үйірлерінің Роджерс жартыторларының алгебралық қасиеттерін зерттеуге арналған. Мұнда айтылған Рождерс жартыторларының, есептелімді функциялар үйірлерінің классикалық Роджерс жартыторларынан айырмашылығын көрсететін элементар қасиеттерін іздеу жалғасуда. $\emptyset' \leq_T A$ болатындай барлық жерде анықталған A-есептелімді функциялар үйірлері қарастырылады. Мақаланың негізгі идеялары [1]-ден алынған.

МИНИМАЛЬНЫЕ ЭЛЕМЕНТЫ И МИНИМАЛЬНЫЕ ПОКРЫТИЯ В ПОЛУРЕШЕТКАХ РОДЖЕРСА

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Ключевые слова: минимальная нумерация, позитивная нумерация, разрешимая нумерация, минимальное покрытие.

Аннотация. Статья посвящена изучению алгебраических свойств полурешеток Роджерса, обобщенно вычислимых семейств всюду определенных функций. На самом деле, это продолжение попыток найти элементарные свойства соответствующих полурешеток Роджерса, отличных от свойств классических полурешеток Роджерса для семейств вычислимых функций. Будут рассматриваться семейства всюду определенных A-вычислимых функций, где $\emptyset' \leq_T A$. Основные идеи взяты из [1].

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