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ANALYTICAL SOLUTION OF THE HEAT EQUATION WITH DISCONTINUOUS COEFFICIENTS AND BOUNDARY FLUX CONDITION

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Abstract. This paper is a continuation of study on analytical solution of heat equation with third type boundary conditions where initial conditions were not considered. Solution of the problem was found analytically.

Introduction. A lengthy paper [1] by D. V. Widder devoted to properties of heat polynomials. Classical methods represented in [2] and [3] are inapplicable for degenerate domains. In this paper we consider analytical solution of the heat equation with discontinuous coefficients in domain with moving boundary which degenerate at the initial time. We follow the method represented in [4] and utilize heat polynomials and integral error functions to solve the problem.

Problem statement. It is required to find the solution of the Heat Equation

$$\frac{\partial u_1}{\partial t} = a_1^2 \frac{\partial^2 u_1}{\partial x^2}, \quad 0 < x < \alpha(t), \qquad \frac{\partial u_2}{\partial t} = a_2^2 \frac{\partial^2 u_2}{\partial x^2}, \quad \alpha(t) < x < \infty \tag{1}$$

with the boundary and initial conditions:

$$-\lambda_1 \frac{\partial u_1(0,t)}{\partial x} = f(t) \tag{2}$$

$$u_2(x,0) = \varphi(x), \tag{3}$$

the conditions of conjugations of temperature and heat flux on a free boundary are

$$u_2(\infty,t) = 0 \tag{4}$$

$$x = \alpha(t): \quad u_1(\alpha(t),t) = u_2(\alpha(t),t) = U_m \tag{5}$$

$$\alpha(0) = 0, \; u_2(\infty,t) = 0 \tag{6}$$

Problem solution. We represent the solution in the following form:

$$\begin{aligned} u_1(x,t) = &\sum_{n=0}^{\infty} A_{2n}(2a_1t)^n \left[i^{2n}erfc\frac{-x}{2a_1\sqrt{t}} + i^{2n}erfc\frac{x}{2a_1\sqrt{t}} \right] + \\ &+\sum_{n=0}^{\infty} A_{2n+1}(2a_1t)^{\frac{2n+1}{2}} \left[i^{2n+1}erfc\frac{-x}{2a_1\sqrt{t}} - i^{2n+1}erfc\frac{x}{2a_1\sqrt{t}} \right] \end{aligned} \tag{7}$$

$$u_2(x,t) = \sum_{n=1}^{\infty} B_n (2a_2t)^{n/2} \left[i^n erfc\frac{-x}{2a_2\sqrt{t}} \right] + \sum_{n=1}^{\infty} C_n (2a_2t)^{n/2} \left[i^n erfc\frac{x}{2a_2\sqrt{t}} \right] \tag{8}$$

where coefficients $A_{2n}, A_{2n+1}, B_n, C_n$ have to be found. Using Hermite polynomials we represent (7) in the form of Heat polynomials:

$$u_1(x,t)=\sum_{n=0}^{\infty}A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m} \tag{9}$$

Making substitution $\sqrt{t}=\tau$

$$u_1(x,t)=\sum_{n=0}^{\infty}A_{2n}\sum_{m=0}^{n}x^{2n-2m}\tau^{2m}\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}\tau^{2m}\beta_{2n+1,m} \tag{10}$$

From (5) for $x=0$

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=0}=\frac{1}{-\lambda_1}P(t)$$

Using above expression we have

$$\sum_{n=0}^{\infty}A_{2n+1}t^n\beta_{2n+1,n}=\frac{1}{-\lambda_1}P(t)$$

$$A_{2n+1}=-\frac{1}{\lambda_1\beta_{2n+1,n}}\cdot\frac{P^{(n)}(0)}{n!} \tag{11}$$

To find A_{2n} we use multinomial coefficients of Newton's Polynomials.

It is known that

$$(x_1+x_2+\ldots+x_m)^n=\sum_{k_1+k_2+\ldots+k_m=n}\binom{n}{k_1,k_2,\ldots,k_{m+1}}\prod_{1\le t\le m}x_t^k$$

where $\binom{n}{k_1,k_2,k_3,\ldots,k_m}=\frac{n!}{k_1!k_2!\ldots k_m!}$ is a *multinomial coefficient*

for

$$\beta(t)=\alpha_1t^{1/2}+\alpha_2t^1+\alpha_{n+1}t^{3/2}+\ldots=\sum_{n=0}^{\infty}\alpha_{n+1}t^{\frac{n+1}{2}} \tag{12}$$

after making substitution $\tau=\sqrt{t}$ we have

$$(\alpha_1\tau+\alpha_2\tau^2+\ldots+\alpha_{m+1}\tau^{m+1})^n=\sum_{k_1+k_2+\ldots+k_{m+1}=n}\binom{n}{k_1,k_2,\ldots,k_{m+1}}\cdot\alpha_1^{k_1}\alpha_2^{k_2}\ldots\alpha_{m+1}^{k_{m+1}}t^{k_1+2k_2+\ldots+(m+1)k_{m+1}} \tag{13}$$

where

$$\binom{n}{k_1,k_2,\ldots,k_{m+1}}\alpha_1^{k_1}\alpha_2^{k_2}\ldots\alpha_{m+1}^{k_{m+1}} \tag{14}$$

is a multinomial coefficient in our case

Thus to derive recurrent formula for A_{2n}, we take both sides of (5), 2k – times derivatives at $\tau=0$, we use multinomial coefficients and get following expressions.

$$\begin{aligned}0\equiv 0^{(4l)}&=\sum_{n=1}^{l}A_{2n}\sum_{m=0}^{n-1}C_{2n,m}[4l]+\sum_{n=l+1}^{2l-1}A_{2n}\sum_{m=0}^{2l-n-1}C_{2n,m+2(n-l)}[4l]\beta_{2n,m+2(n-l)}+A_{4l}\beta_{4l,2l}+\\&+\sum_{n=1}^{l}A_{2n-1}\sum_{m=0}^{n-1}C_{2n-1,m}[4l]\beta_{2n-1,m}+\sum_{n=l+1}^{2l}A_{2n-1}\sum_{m=0}^{2l-n}C_{2n-1,m+2(n-l)-1}[4l]\beta_{2n-1,m+2(n-l)-1}\end{aligned} \tag{15}$$

where l=1,2,… and