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SOME PROPERTIES OF THE NONCOMMUTATIVE $H_p^{(r,s)}(A;l_\infty)$ AND $H_q(A;l_1)$ SPACES

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Abstract. In this paper we introduced noncommutative vector-valued $H_p(A;l_1)$ and $H_p(A,\ell_\infty)$ spaces. Then we proved two criteria for positive sequences and proved the analogue of Saito's theorem on $H_p(A;l_1)$ and $H_p(A,\ell_\infty)$ spaces.

Introduction. Let M be a finite von Neumann algebra equipped with a faithful normal normalized tracial state τ. Let N be a von Neumann subalgebra of M, and let $\Phi: M \to N$ be the unique normal faithful conditional expectation such that $\tau\circ\Phi=\tau$. A finite subdiagonal algebra of M with respect to Φ is a w*-closed subalgebra A of M satisfying the following conditions:

(i) $A+A^*$ is weak* dense in M;

(ii) Φ is multiplicative on A, i.e. $\Phi(ab)=\Phi(a)\Phi(b)$, for all $a,b\in M$;

(iii) $A\cap A^*=N$, where A^* is the family of all adjoint elements of the element of A, i.e., $A^*=\{a^*: a\in A\}$.

The algebra N is called the diagonal of A. It's proved by Exel in [16] that a finite subdiagonal algebra A is automatically maximal in the sense that, if B is another subdiagonal algebra with respect to Φ containing A, then $B=A$. This maximality yields the following useful characterization of A:

$$A=\{x\in M: \tau(xc)=0, \forall c\in A_0\} \quad (1.1)$$

where $A_0=A\cap\ker\Phi$ (see [1]).

Given $0<p\le\infty$ we denote by $L_p(M)$ the usual noncommutatve L_p-spaces associated with $(M;\tau)$. Recall that $L_\infty(M)=M$ equipped with the operator norm (see [15]). The norm of $L_p(M)$ will be denoted by $\|\bullet\|_p^p=\tau(|\bullet|^p)$. For $0<p\le\infty$ we define $H_p(A)$ to be closure of A in $L_p(M)$ and for $p=\infty$ we simply set $H_\infty(A)=A$ for convenience. These are so called Hardy spaces associated with A. They are noncommutative extensions of the classical Hardy space on the torus T. We refer to [1] and [15] for more examples. The theory of vector-valued noncommutative L_p-spaces are introduced by Pisier in [14]. Pisier considered the case M is hyperfinite and later by Junge in [12] (see also [13]) for the general case. We refer the reader notably to the recent work by Defant/Junge [7]. Junge and Xu introduced the spaces $L_p^{(r,s)}(M;l_\infty)$ and $L_p(M;l_1)$ spaces (see also [9]). They proved that both spaces $L_p^{(r,s)}(M;l_\infty)$ and

$L_p(M;l_1)$ spaces are Banach spaces for $1 \le p < \infty$. More precisely, let $1 \le p < \infty$ such that $1/p + 1/q = 1$. Then the duality theorem holds, i.e. $L_p(M;l_1)^* = L_q(M;l_\infty)$ (1.2). We now define the analogue of $L_p(M;l_\infty)$ and $L_p(M;l_1)$ spaces by a similar way (see [4]).

Definition 1.1 Let $0 < p \le \infty$.

(i) We define $H_p(A,\ell_\infty)$ to be the space of all $x = \{x_n\}_{n\ge1}$ in $H_p(A)$ for which there exist $a, b \in H_{2p}(A)$ and a bounded sequence $\{y_n\}_{n\ge1} \subset H_\infty(A)$ such that $x_n = ay_nb$, for all $n \ge 1$. Given $x \in H_p(A,\ell_\infty)$, define

$$\|x\|_{H_p(A;l_\infty)} = \inf\left\{\|a\|_{2p} \sup_{n\ge1}\|y_n\|_\infty \|b\|_{2p}\right\},$$

where the infimum runs over all factorizations of $x = \{x_n\}_{n\ge1}$ as above.

(ii) We define $H_p(A,\ell_1)$ as the space of all sequence in $x = \{x_n\}_{n\ge1} \subset H_p(A)$ which can be decomposed as $x_n = \sum_{k\ge1} u_{kn}v_{nk}$, for al $n \ge 1$, for two families $\{u_{kn}\}_{k,n\ge1}$ and $\{v_{kn}\}_{k,n\ge1}$ in $H_{2p}(A)$ such that $\sum_{k,n\ge1} u_{kn}u_{kn}^*, \sum_{k,n\ge1} v_{nk}^*v_{nk} \in L_p(M)$.

In this space we define norm the following form:

$$\|x\|_{H_p(A;l_1)} = \inf\left\{\left\|\sum_{k,n\ge1} u_{kn}u_{kn}^*\right\|_p^{1/2} \cdot \left\|\sum_{k,n\ge1} v_{nk}^*v_{nk}\right\|_p^{1/2}\right\},$$

where the infimum runs over all decompositions of $x = \{x_n\}_{n\ge1}$ as above .

In [4] it is proved that both $H_p(A;l_1)$ and $H_p(A,\ell_\infty)$ spaces are quasi-Banach spaces. Then proved some basic properties such as duality theorem and contractibility of conditional expectations on these spaces.

Formula (1.1) admits the following $H_p(A)$ analogue proved by Saito in [18]:

$$H_p(A) = \{x \in L_p(M) : \tau(xc) = 0, \forall c \in A_0\},\ 1 \le p < \infty \quad (1.3)$$

Then in [3] Bekjan and Xu proved that formula (1.3) holds for every $0 < p \le \infty$. This noncommutative Hardy spaces have received a lot of attention since Arveson's pioneer work. We refer the reader a series of newly finished papers by Blecher/Labuschagne [2],[5],[6], whereas more references on previous works can be found in the survey paper [15]. Most results on the classical Hardy spaces on the torus have been established in this noncommutative setting. Here we mention some of them directly related with the objective of this paper. The main purpose of the present paper is to extend formula (1.1) to the spaces $H_p(A;l_1)$ and $H_p(A,\ell_\infty)$.

2. **Main results.** To gain a very first understanding on $H_p(A;l_1)$ and $H_p(A,\ell_\infty)$ space above we define, following Lemmas are very useful.

Lemma 2.1. Let $1 \le p \le \infty$. Then for any $x = \{x_n\}_{n\ge1} \in H_p(A;l_1)$ we have

$$\left\|\sum_{n=1}^{\infty} x_n\right\|_{H_p(A)} \le \|x\|_{H_p(A;l_1)}$$

In addition a positive sequence $x = \{x_n\}_{n\ge1}$ (i.e. $x_n \ge 0$, $\forall\ n \ge 1$) belongs to $H_p(A;l_1)$ if and only if

$$\sum_{n=1}^{\infty} x_n \in H_p(A)$$

Proof. Let $x_n = \sum_{k\geq 1} u_{kn} v_{nk}$ is a decomposition of x, then by the Hölder inequality in Proposition 3.6 in [18],

$$\left\| \sum_{n=1}^{\infty} x_n \right\|_{H_p(A)} \leq \left\| \sum_{k,n=1}^{\infty} u_{kn} \right\|_p \cdot \left\| \sum_{n,k=1}^{\infty} v_{nk} \right\|_p ;$$

whence

$$\left\| \sum_{n=1}^{\infty} x_n \right\|_{H_p(A)} \leq \|x\|_{H_p(A;l_1)}.$$

In fact, this inequality holds for any $x \in H_p(A;l_1)$. To prove converse for a positive x we need only to take $\{u_{kn}\}_{k,n\geq 1}$ and $\{v_{kn}\}_{k,n\geq 1}$ defined by $u_{kn} = v_{nk} = x_n^{1/2}$ if $n = k$, $u_{kn} = v_{nk} = 0$ otherwise.

Lemma 2.2. A positive sequence $x = \{x_n\}_{n\geq 1}$ belongs to $H_p(A, \ell_\infty)$ if and only if there exists positive $a \in H_p(A)$ such that

$$x_n \leq a, \ \forall \ n \geq 1.$$

Proof. Let $\{x_n\}_{n\geq 1} \subset H_p(A)$. Assume that there exists positive $a \in H_p(A)$ such that $x_n \leq a$, $\forall \ n \geq 1$. Then by Remark 2.3 in [7] there exists a contraction operator $u_n \in M$ such that $x_n^{1/2} = u_n a^{1/2}$, so $x_n = a^{1/2} u_n^* u_n a^{1/2}$. Thus $x \in L_p(M;l_\infty)$ and $\|x\|_{L_p(M;l_\infty)} \leq \|a\|_p$ (see [9]). Then by using Proposition 2.1 in [4], we obtain $x \in H_p(A;l_\infty)$. On the other hand if $x \in H_p(A;l_\infty)$ is positive, then for all $n \geq 1$ we can find a positive $a \in H_p(A)$ and positive contractions $y_n \in A$ such that $x_n = a^{1/2} y_n a^{1/2}$. From this it is easy to show that $x_n \leq a$, which is the conclusion.

The following Theorems are the analogue of Saito's theorem (see [18]) on noncommutative $H_p(A;l_1)$ and $H_p(A, \ell_\infty)$ spaces.

Theorem 2.1. Let $1 \leq p < \infty$. Then we have the following, where $H_p^0(A;l_\infty) = \{x \in L_p(M;l_\infty) : \Phi(x_n) = 0, \forall c \in A \ and \ \forall n \geq 1\}$:

$$H_p(A;l_\infty) = \{x \in L_p(M;l_\infty) : \Phi(x_n) = 0, \forall c \in A_0 \ and \ \forall n \geq 1\} \quad (2.1)$$

Moreover,

$$H_p^0(A;l_\infty) = \{x \in L_p(M;l_\infty) : \Phi(x_n) = 0, \forall c \in A \ and \ \forall n \geq 1\} \quad (2.2)$$

Proof. The inclusion $H_p(A;l_\infty) \subset \{x \in L_p(M;l_\infty) : \Phi(x_n) = 0, \forall c \in A_0 \ and \ \forall n \geq 1\}$ is clearly. Let $y \in \{x \in L_p(M;l_\infty) : \Phi(x_n) = 0, \forall c \in A_0 \ and \ \forall n \geq 1\}$. Then by Lemma 2.1 (i) in [4] there exist $a, b \in H_{2p}(A)$ and a bounded sequence $\{z_n\}_{n\geq 1} \subset M$ such that $x_n = a y_n b$, for all $n \geq 1$, where $a^{-1}, b^{-1} \in A$ and $\sup_n \|z_n\|_\infty \leq 1$. On the other hand we have $\tau(xc) = 0, \ \forall c \in A_0$. Since $a^{-1} s b^{-1} \in A_0, \ \forall s \in A_0$, substituting c by $a^{-1} s b^{-1}$ we obtain $z_n \in A$ (see [18]), so $\{y_n\}_{n\geq 1} \in H_p(A;l_\infty)$. Now we prove the (2.2).It is obvious that $H_p^0(A;l_\infty) \subset \{x \in L_p(M;l_\infty) : \Phi(x_n) = 0, \forall c \in A \ and \ \forall n \geq 1\}$.

Let $x \in L_p(M;l_\infty)$, then as above by using Lemma 2.1 (i) in [4] and since $\tau(x_n d) = 0,\ \forall d \in A_0$ we get that $x \in H_p(A;l_\infty)$. On the other hand we have. Then since $x_n \in L_p(M),\ \forall n \geq 1$, we deduce $x_n \in H_p^0(A),\ \forall n \geq 1$ (see [18]), which is the conclusion.

Similar to Theorem 2.1 and by using Lemma 2.1 (ii) in [4] we have the following result.

Theorem 2.2. Let $1 \leq p < \infty$. Then we have the following, where $H_p^0(A;l_1) = \{x \in L_p(M;l_1) : \Phi(x_n) = 0, \forall c \in A \text{ and } \forall n \geq 1\}$:

$$H_p(A;l_1) = \{x \in L_p(M;l_1) : \Phi(x_n) = 0, \forall c \in A_0 \text{ and } \forall n \geq 1\} \quad (2.1)$$

Moreover,

$$H_p^0(A;l_1) = \{x \in L_p(M;l_1) : \Phi(x_n) = 0, \forall c \in A \text{ and } \forall n \geq 1\} \quad (2.2)$$

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КОММУТАТИВТІ ЕМЕС $H_p(A;l_1)$ ЖӘНЕ $H_p(A,\ell_\infty)$ КЕҢІСТІКТЕРІНІҢ КЕЙБІР ҚАСИЕТТЕРІ

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Тірек сөздер: фон Нейман алгебрасы, математикалық күтім, субдиагоналды алгебра, τ-өлшемді опера-*торлар, коммутативті емес вектор мәнді Харди кеңістіктері.

Аннотация. Мақалада коммутативті емес $H_p(A;l_1)$ және $H_p(A,\ell_\infty)$ кеңістіктері қарастырылған $H_p(A;l_1)$ және $H_p(A,\ell_\infty)$ кеңістіктерінде оң операторлар тізбегі үшін екі критерий және Сайтоның теоремасына ұқсас теорема дәлелденген.

НЕКОТОРЫЕ СВОЙСТВА НЕКОММУТАТИВНЫХ ПРОСТРАНСТВ $H_p(A;l_1)$ И $H_p(A,\ell_\infty)$

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Ключевые слова: алгебра фон Неймана, условное ожидание, субдиагональная алгебра, τ-измеримые операторы, некоммутативные векторнозначные пространства Харди.

Аннотация. В этой работе рассмотрена некоммутативные пространства $H_p(A;l_1)$ и $H_p(A,\ell_\infty)$. Доказаны два критерия для положительных последовательности оператора и аналог теоремы Сайто на пространстве $H_p(A;l_1)$ и $H_p(A,\ell_\infty)$

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