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ON THE SOLVING OF INITIAL-BOUNDARY VALUE PROBLEM
FOR SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS
OF THE THIRD ORDER

Abstract. The initial-boundary value problem for the special system of the third-order partial differential
equations is considered. We study the existence of classical solutions to initial-boundary value problem for the
special system of the third-order partial differential equations and offer the methods for constructing their
approximate solutions. Sufficient conditions for the existence and uniqueness of classical solution to initial-boundary
value problem for the system of the third order partial differential equations are established. By introduction of new
unknown function, we have reduced the considered problem to an equivalent problem consisting of a nonlocal
problem for the system of hyperbolic equations of the second order with functional parameter and an integral
relation. We have offered the algorithm to find an approximate solution to the investigated problem and have proved
its convergence. Sufficient conditions for the existence of unique solution to the equivalent problem with parameter
are established. Conditions of unique solvability to the initial-boundary value problem for the system of partial
differential equations of the third order are obtained in the terms of initial data.

Key Words: partial differential equation of the third order, initial-boundary value problem, nonlocal problem,
system of hyperbolic equations, solvability, algorithm.

1. Introduction. In recent decades, there has been a great interest to initial-boundary value problems
for partial differential equations and systems of the third order. This is due to the appearance of such
problems in the mathematical modeling of various natural science processes [1-6]. Quite a number of
works devoted to the investigation of various problems for partial differential equations of the third order
with two independent variables, bibliography and analysis can be seen in [1,2,5]. The system of partial
differential equations of the third order began to be studied relatively recently [5]. In the present work, we
consider the special system of partial differential equations of the third order at a rectangular domain. The
boundary condition for time variable is specified as a combination of values from the partial derivatives of
the required solution on the first and the second orders by spatial variable. We investigate the questions of
existence and uniqueness of the classical solution to initial-boundary value problem for system of partial
differential equations of the third order and its applications.

Methods. To solve the the considered problem we use a method of introduction additional functional
parameters [7-25]. The original problem is reduced to an equivalent problem consisting from nonlocal
problem for a system of hyperbolic equations of the second order with functional parameters and integral
relations. Sufficient conditions of the unique solvability to investigated problem are established in the
terms of initial data. Algorithms of finding a solution to the equivalent problem are constructed.
Conditions of unique solvability to initial-boundary value problem for a system of partial differential
equations of the third order are established in the terms of coefficient of system and boundary matrices.
2. Statement of problem. At the domain \( \Omega = [0, T] \times [0, \omega] \) we consider the following initial-boundary value problem for the special system of partial differential equations

\[
\frac{\partial^3 u}{\partial t \partial x^2} = A(t, x)u + f(t, x), \quad (t, x) \in \Omega ,
\]

\[
K_0(x) \frac{\partial^2 v(t_0, x)}{\partial x^2} + L_0(x) \frac{\partial v(t_0, x)}{\partial t} \bigg|_{t=t_{-0}} + M_0(x) \frac{\partial v(t_0, x)}{\partial x} + P_0(x) u(t_0, x) + K_1(x) \frac{\partial^2 u(t_1, x)}{\partial x^2} +
\]

\[
+ L_1(x) \frac{\partial u(t_1, x)}{\partial x} \bigg|_{t=t_{-1}} + M_1(x) \frac{\partial u(t_1, x)}{\partial x} + P_1(x) u(t_1, x) + K_2(x) \frac{\partial^2 u(t_2, x)}{\partial x^2} + L_2(x) \frac{\partial u(t_2, x)}{\partial x} \bigg|_{t=t_{-2}} +
\]

\[
+ M_2(x) \frac{\partial u(t_2, x)}{\partial x} + P_2(x) u(t_2, x) = \phi(x), \quad x \in [0, \omega],
\]

where \( u(t, x) = col(u_1(t, x), u_2(t, x), ..., u_n(t, x)) \) is an unknown function, the \( n \times n \)-matrix \( A(t, x) \) and \( n \)-vector function \( f(t, x) \) are continuous on \( \Omega \), the \( n \times n \)-matrices \( K_i(x), L_i(x), M_i(x), P_i(x) \) and \( n \)-vector-function \( \phi(x) \) are continuous on \([0, \omega] \), \( i = 0, 1, 2 \), \( 0 \leq t_0 < t_1 < t_2 \leq T \), the \( n \)-vector-functions \( \psi_j(t) \) and \( \psi_1(t) \) are continuously differentiable on \([0, T] \). The initial data satisfy the condition of approval.

A function \( u(t, x) \in C(\Omega, R^n) \) having partial derivatives \( \frac{\partial u(t, x)}{\partial x} \in C(\Omega, R^n) \), \( \frac{\partial u(t, x)}{\partial t} \in C(\Omega, R^n) \), \( \frac{\partial^2 u(t, x)}{\partial x^2} \in C(\Omega, R^n) \), \( \frac{\partial^2 u(t, x)}{\partial t^2} \in C(\Omega, R^n) \), \( \frac{\partial^3 u(t, x)}{\partial x^2 \partial t} \in C(\Omega, R^n) \) is called a classical solution to the problem (1)--(4) if it satisfies system (1) for all \( (t, x) \in \Omega \), and boundary conditions (2), (3) and (4).

We investigate the questions of existence and uniqueness of the classical solutions to the initial-boundary value problem for the system of partial differential equations of the third order (1)--(4) and the approaches to constructing its approximate solutions. For this goals, we applied the method of introduction of the additional functional parameters proposed in [7-25] for solving nonlocal boundary value problems for systems of hyperbolic equations with mixed derivatives. Considered problem is provided to nonlocal problem for system of hyperbolic equations of the second order including additional function and integral relation. The algorithm of finding the approximate solution of the investigated problem is proposed and its convergence proved. Sufficient conditions of the existence unique classical solution to the problem (1)--(4) are obtained in the terms of initial data.

3. Scheme of the method and reduction to equivalent problem.

We introduce a new unknown function \( v(t, x) = \frac{\partial u(t, x)}{\partial x} \) and write the problem (1)--(4) in the following form

\[
\frac{\partial^2 v}{\partial t \partial x} = A(t, x)v + f(t, x), \quad (t, x) \in \Omega ,
\]

\[
K_0(x) \frac{\partial v(t_0, x)}{\partial x} + L_0(x) \frac{\partial v(t_0, x)}{\partial t} \bigg|_{t=t_{-0}} + M_0(x) v(t_0, x) + K_1(x) \frac{\partial v(t_1, x)}{\partial x} + L_1(x) \frac{\partial v(t_1, x)}{\partial t} \bigg|_{t=t_{-1}} +
\]

\[
+ M_1(x) v(t_1, x) + K_2(x) \frac{\partial v(t_2, x)}{\partial x} + L_2(x) \frac{\partial v(t_2, x)}{\partial t} \bigg|_{t=t_{-2}} + M_2(x) v(t_2, x) +
\]

\[
+ P_0(x) u(t_0, x) + P_1(x) u(t_1, x) + P_2(x) u(t_2, x) = \phi(x), \quad x \in [0, \omega] .
\]
\( v(t, 0) = \psi_\alpha(t), \quad t \in [0, T], \) \hfill (7)

\[ u(t, x) = \psi_\beta(t) + \int_0^x v(t, \xi) d\xi, \quad (t, x) \in \Omega. \] \hfill (8)

Here the condition (3) is taken into account in (8).

A pair functions \( (v(t, x), u(t, x)) \), where the function \( v(t, x) \in C(\Omega, R^n) \) has partial derivatives

\[ \frac{\partial v(x, t)}{\partial x} \in C(\Omega, R^n), \quad \frac{\partial v(x, t)}{\partial t} \in C(\Omega, R^n), \quad \frac{\partial^2 v(x, t)}{\partial x \partial t} \in C(\Omega, R^n), \] the function \( u(t, x) \in C(\Omega, R^n) \) has partial derivatives

\[ \frac{\partial u(x, t)}{\partial x} \in C(\Omega, R^n), \quad \frac{\partial u(x, t)}{\partial t} \in C(\Omega, R^n), \quad \frac{\partial^2 u(x, t)}{\partial x^2} \in C(\Omega, R^n), \]

\[ \frac{\partial^2 u(x, t)}{\partial t^2} \in C(\Omega, R^n), \quad \frac{\partial^3 u(x, t)}{\partial x \partial t^2} \in C(\Omega, R^n), \]

is called a solution to the problem (5)–(8) if it satisfies the system of hyperbolic equations (5) for all \((t, x) \in \Omega\), the boundary conditions (6), (7), and the integral relation (8).

At fixed \( u(t, x) \) the problem (5)–(7) is the nonlocal problem for the system of hyperbolic equations with respect to \( v(t, x) \) on \( \Omega \). The integral relation (8) allows us to determine the unknown function \( u(t, x) \) for all \((t, x) \in \Omega\).

4. Algorithm. The unknown function \( v(t, x) \) will be determined from the nonlocal problem for the system of hyperbolic equations (5)–(7). The unknown function \( u(t, x) \) will be found from integral relation (8).

If we know the function \( u(t, x) \), then from the nonlocal problem (5)–(7) we find the function \( v(t, x) \). Conversely, if we know the function \( u(t, x) \), then from the nonlocal problem (5)–(7) we find the function \( v(t, x) \). Since the functions \( u(t, x) \) and \( v(t, x) \) are unknown together for finding of the solution to the problem (5)–(8) we use an iterative method. The solution to the problem (5)–(8) is the pair functions \( (v^m(t, x), u^m(t, x)) \) we defined as a limit of a sequence of pairs \( (v^{(m)}(t, x), u^{(m)}(t, x)) \), \( m = 0, 1, 2, \ldots \), according to the following algorithm:

**Step 0.** 1) Suppose in the right-hand part of the system (5) \( u(t, x) = \psi_\alpha(t) \), from the nonlocal problem (5)–(7) we find the initial approximation \( v^{(0)}(t, x) \) for all \((t, x) \in \Omega\);

2) From the integral relation (8) under \( v(t, x) = v^{(0)}(t, x) \), we find the function \( u^{(0)}(t, x) \), for all \((t, x) \in \Omega\).

**Step 1.** 1) Suppose in the right-hand part of the system (5) and boundary condition (6) \( u(t, x) = u^{(5)}(t, x) \), from the nonlocal problem (5)–(7) we find the first approximation \( v^{(1)}(t, x) \) for all \((t, x) \in \Omega\).

2) From the integral relation (8) under \( v(t, x) = v^{(1)}(t, x) \), we find the function \( u^{(1)}(t, x) \) for all \((t, x) \in \Omega\).

And so on.

**Step \( m \).** 1) Suppose in the right-hand part of the system (5) and boundary condition (6) \( u(t, x) = u^{(m-1)}(t, x) \), \( \tilde{v}(t, x) = v^{(m-1)}(t, x) \), from the nonlocal problem (5)–(7) we find the \( m \)-the approximation \( v^{(m)}(t, x) \) for all \((t, x) \in \Omega:\)

\[ \frac{\partial^2 v^{(m)}(t, x)}{\partial t \partial x} = A(t, x) u^{(m-1)}(t, x) + f(t, x), \quad (t, x) \in \Omega, \] \hfill (9)
\[ K_0(x) \frac{\partial \nu^{(m)}(t_0, x)}{\partial t} + L_0(x) \frac{\partial \nu^{(m)}(t, x)}{\partial t} \Bigg|_{t=t_0} + M_0(x) \nu^{(m)}(t_0, x) + K_1(x) \frac{\partial \nu^{(m)}(t_1, x)}{\partial t} + L_1(x) \frac{\partial \nu^{(m)}(t, x)}{\partial t} \Bigg|_{t=t_1} + \\
+ M_2(x) \nu^{(m)}(t_2, x) + K_2(x) \frac{\partial \nu^{(m)}(t_2, x)}{\partial t} + L_2(x) \frac{\partial \nu^{(m)}(t, x)}{\partial t} \Bigg|_{t=t_2} + M_2(x) \nu^{(m)}(t_2, x) + \\
+ P_0(x) u^{(m-1)}(t_0, x) + P_1(x) u^{(m-1)}(t_1, x) + P_2(x) u^{(m-1)}(t_2, x) = \varphi(x), \ x \in [0, \omega], \]

\[ \nu^{(m)}(t, 0) = \psi_1(t), \ t \in [0, T]. \]

\[ u^{(m)}(t, x) = \psi_1(t) + \int_0^x \nu^{(m)}(t, \xi) d\xi, \ (t, x) \in \Omega. \]

5. The main result.

The following theorem gives conditions of feasibility and convergence of the constructed algorithm and the conditions of the existence unique solution to the problem (5)--(8).

**Theorem 1.** Suppose that

i) the \( n \times n \)-matrix \( A(t, x) \) and \( n \)-vector function \( f(t, x) \) are continuous on \( \Omega \);

ii) the \( n \times n \)-matrices \( K_i(x) \), \( L_i(x) \), \( M_i(x) \), \( P_i(x) \) and \( n \)-vector-function \( \varphi(x) \) are continuous on \( [0, \omega], \ i = 0, 1, 2; \)

iii) the \( n \)-vector-functions \( \psi_0(t) \) and \( \psi_1(t) \) are continuously differentiable on \( [0, T] \);

iv) the \( n \times n \)-matrix \( Q(x) = K_0(x) + K_1(x) + K_2(x) \) is invertible for all \( x \in [0, \omega] \).

Then nonlocal problem for the system of hyperbolic equations with integral relation (5)--(8) has a unique solution.

**Theorem 2.** Suppose that the conditions i) - iv) of Theorem 1 are fulfilled.

Then the initial-boundary value problem for system of partial differential equations of the third order (1)--(4) has a unique classical solution.

The proof of the theorems is similar to the scheme of the proof of theorems [20].

6. Special case. Now, we consider a special initial-boundary value problem for the system of partial differential equations

\[ \frac{\partial^2 u}{\partial t^2} = A(t, x) u + f(t, x), \ (t, x) \in \Omega, \]

\[ S_0(x) u(t_0, x) + S_1(x) u(t_1, x) + S_2(x) u(t_2, x) = \varphi(x), \ x \in [0, \omega], \]

\[ u(t, 0) = \psi_0(t), \ t \in [0, T]. \]

\[ \frac{\partial u(t, x)}{\partial x} \Bigg|_{x=0} = \psi_1(t), \ t \in [0, T]. \]

For this case, the matrix \( Q(x) \) is not invertible. Therefore, we will additionally assume that the \( n \times n \)-matrices \( S_i(x) \) and the \( n \)-vector function \( \varphi(x) \) are twice continuously differentiable on \( [0, \omega], \ i = 0, 1, 2 \). In addition, the compatibility conditions of the initial data are fulfilled:

\[ S_0(0) \psi_0(t_0) + S_1(0) \psi_0(t_1) + S_2(0) \psi_0(t_2) = \varphi(0), \]

\[ \hat{S}_0(0) \psi_0(t_0) + \hat{S}_1(0) \psi_0(t_1) + \hat{S}_2(0) \psi_0(t_2) + S_0(0) \psi_1(t_0) + S_1(0) \psi_1(t_1) + S_2(0) \psi_1(t_2) = \hat{\varphi}(0). \]
Using the properties of matrices $S_i(x)$ and $\varphi(x)$ we twice differentiate relation (14) with respect to $x$. We have

$$
S_0(x) \frac{\partial^2 u(t_0, x)}{\partial x^2} + S_1(x) \frac{\partial^2 u(t_1, x)}{\partial x^2} + S_2(x) \frac{\partial^2 u(t_2, x)}{\partial x^2} + 2S_0(x) \frac{\partial u(t_0, x)}{\partial x} + 2S_1(x) \frac{\partial u(t_1, x)}{\partial x} + 2S_2(x) \frac{\partial u(t_2, x)}{\partial x} + 2\dot{S}_0(x) \frac{\partial u(t_0, x)}{\partial x} + 2\dot{S}_1(x) \frac{\partial u(t_1, x)}{\partial x} + 2\dot{S}_2(x) \frac{\partial u(t_2, x)}{\partial x} = \varphi(x), \ x \in [0, \omega].
$$

(17)

We obtain the original problem (1)–(4) again, where $K_i(x) = S_i(x)$, $L_i(x) = 0$, $M_i(x) = 2\dot{S}_i(x)$, $P_i(x) = \ddot{S}_i(x)$, $i = 0, 1, 2$.

The following assertion is valid.

**Theorem 3.** Suppose that

i) the $n \times n$-matrix $A(t, x)$ and $n$-vector function $f(t, x)$ are continuous on $\Omega$;

ii) the $n \times n$-matrices $S_i(x)$ and $n$-vector-function $\varphi(x)$ are twice continuously differentiable on $[0, \omega]$, $i = 0, 1, 2$;

iii) the $n$-vector-functions $\psi_0(t)$ and $\psi_1(t)$ are continuously differentiable on $[0, T]$;

iv) the compatibility conditions of the initial data are fulfilled: $S_0(0)\psi_0(t_0) + S_1(0)\psi_1(t_1) + S_2(0)\psi_2(t_2) = \varphi(0)$,

$\dot{S}_0(0)\psi_0(t_0) + \dot{S}_1(0)\psi_1(t_1) + \dot{S}_2(0)\psi_2(t_2) + S_0(0)\psi_1(t_0) + S_0(0)\psi_1(t_1) + S_0(0)\psi_1(t_2) = \varphi(0)$;

v) the $n \times n$-matrix $\dot{Q}(x) = S_0(x) + S_1(x) + S_2(x)$ is invertible for all $x \in [0, \omega]$.

Then initial-boundary value problem for the system of partial differential equations of the third order (13)–(16) has a unique classical solution.

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О РЕШЕНИИ НАЧАЛЬНО-КРАЕВОЙ ЗАДАЧИ ДЛЯ СИСТЕМЫ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В ЧАСТНЫХ ПРОИЗВОДНЫХ ТРЕТЬЕГО ПОРЯДКА

Аннотация. Рассматривается начальная-краевая задача для специальной системы дифференциальных уравнений в частных производных третьего порядка. Исследуются вопросы существования классического решения начальной-краевой задачи для специальной системы дифференциальных уравнений в частных производных третьего порядка и предлагаются методы построения их приближенных решений. Установлены достаточные условия существования и единственности классического решения системы дифференциальных уравнений в частных производных третьего порядка. Путем введения новой неизвестной функции исследуемая задача сводится к эквивалентной задаче, состоящей из нелокальной задачи для системы гиперболических уравнений второго порядка с функциональным параметром и интегрального соотношения. Предложены алгоритмы нахождения приближенного решения исследуемой задачи и доказана их сходимость. Установлены достаточные условия существования единственного решения эквивалентной...
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УШІНШІ РЕТТЕДЕРЕС ТУЫНДЫЛЫ ДИФФЕРЕНЦИАЛДЫҚ
ТЕҢДЕУЛЕР ЖУЙЕСІ УШІН БАСТАПҚЫ-ШЕТТІК ЕСЕПТІҢ ШЕШІМІ ТУРАЛЫ

Аннотация. Ушінші ретті дербес түйінділы дифференциалдық тәндеулердің арнайы жүйесі үшін бастапқы-шеттік есеп қарады жұмысы. Ушінші ретті дербес түйінділы дифференциалдық тәндеулердің арнайы жүйесі үшін бастапқы-шеттік есептің классикалық шешімінің бар болуы мүмкін екен. Шешім үшін дәрежелі құрылыştırma құрылысқа жатады. Ушінші ретті дербесті түйінділы дифференциалдық тәндеулер үшін бастапқы-шеттік есептің классикалық шешімінің бар болуы мен жалпылымдық жетілікті шарттарға қаіымдастырылған. Жаңа есептің қасиетін таңдай алғанда есеп үшін дифференциалдық тәндеулердің арнайы жүйесі үшін параметрлер өзгерту және ізгеірлік құрама тәріздік пара-пар есепте қалып тартілған. Зерттелген есептің шешімі бойынша алғашқы дәрежелі құрылысқа қаіымдастырылған. Параметрлер үшін дифференциалдық тәндеулердің арнайы жүйесі үшін бастапқы-шеттік есептің бірнеше шешімділігінің шарттарға бериш сапаты жасалып, термінінде алынған.

Түйін сөздер: ушінші ретті дербесті түйінділы дифференциалдық тәндеу, бастапқы-шеттік есеп, қалыңдық кесте, дифференциалдық тәндеулер, шешімдер, алгоритм.

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